# Testing Random Numbers: Theory and Practice

Prof. Michael Mascagni

Applied and Computational Mathematics Division, Information Technology Laboratory
National Institute of Standards and Technology, Gaithersburg, MD 20899-8910 **USA**
AND
Department of Computer Science
Department of Mathematics
Department of Scientific Computing
Graduate Program in Molecular Biophysics
Florida State University, Tallahassee, FL 32306 **USA**

E-mail: mascagni@fsu.edu or mascagni@nist.gov
URL: http://www.cs.fsu.edu/∼mascagni

# Overview

- Chi-Square Test
- The Kolmogorov-Smirnov (K-S)Test
- Empirical Tests
  - Equidistribution Test (Frequency Test)
  - Serial Test
  - Gap Test
  - Poker Test
  - Coupon Collector's Test
  - Permutation Test
  - Runs Test
  - Maximum of t Test
  - Collision Test
  - Serial Correlation Test
- The Spectral Test

# Chi-Square Test

Eg. "Throwing 2 dice"

$s$ : Value of the sum of the 2 dice.
$p_s$ : Probability.

| $s$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_s$ | $\frac{1}{36}$ | $\frac{1}{18}$ | $\frac{1}{12}$ | $\frac{1}{9}$ | $\frac{5}{36}$ | $\frac{1}{6}$ | $\frac{5}{36}$ | $\frac{1}{9}$ | $\frac{1}{12}$ | $\frac{1}{18}$ | $\frac{1}{36}$ |

If we throw dice 144 times:

| $s$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Observed: $Y_s$ | 2 | 4 | 10 | 12 | 22 | 29 | 21 | 15 | 14 | 9 | 6 |
| Expected: $np_s$ | 4 | 8 | 12 | 16 | 20 | 24 | 20 | 16 | 12 | 8 | 4 |

# Chi-Square Test

Is a pair of dice loaded?
We can't make a definite yes/no statement, but we can give a probabilistic answer.
We can form the Chi-Square Statistic.

$$\chi^2 = \sum_{1 \le s \le k} \frac{(Y_s - np_s)^2}{np_s}$$

$$= \frac{1}{n} \sum_{1 \le s \le k} \left( \frac{Y_s^2}{p_s} \right) - n$$

$\chi^2 = k - 1$: degrees of freedom

$k$: Number of categories
$n$: Number of observances

# Table of Chi-Square Distribution

Entry in row $\chi^2$ under column $p$ is $x$, which means

"The quantity $\chi^2$ will be less than or equal to $x$,
with approximate probability $p$, if $n$ is large enough."

Example:

| Value of s | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Experiment 1, $Y_s$ | 4 | 10 | 10 | 13 | 20 | 18 | 18 | 11 | 13 | 14 | 13 |
| Experiment 2, $Y_s$ | 3 | 7 | 11 | 15 | 19 | 24 | 21 | 17 | 13 | 9 | 5 |

$$\chi_1^2 = 29\frac{59}{120} \qquad \chi_2^2 = 1\frac{11}{120}$$

# Table of Chi-Square Distribution

$$\chi_1^2 = 29\frac{59}{120} \qquad \chi_2^2 = 1\frac{11}{120}$$

Discussion:

$\chi_1^2$ is too high, $\chi^2$ 0.1% of the time.
$\chi_2^2$ is too low, $\chi^2$ 0.01% of the time.

Both represent $x$ with a significant departure from randomness.

To use Chi-Square distribution table, $n$ should be large.
How large should $n$ be?

Rule of thumb:
$n$ should be large enough to make each $np_s$ be 5 or greater.

# Chi-Square Test

1. Large number *n* of independent observations.
2. Count the number of observations on *k* categories.
3. Compute $\chi^2$.
4. Look up Chi-Square distribution table.

| | |
|---|---|
| $\chi^2$<1% or $\chi^2$>99% | reject |
| 1%<$\chi^2$<5% or 95%<$\chi^2$<99% | suspect |
| 5%<$\chi^2$<10% or 90%<$\chi^2$<95% | almost suspect |
| otherwise | accept |

# The Kolmogorov-Smirnov (K-S) Test

| | | |
|---|---|---|
| $\chi^2$ Test | : | for discrete random data |
| K-S Test | : | for continuous random data |

Def: $F(x) = P[X \leq x)]$, cumulative distribution function (CDF) for r.v. $X$

$n$ independent observations of $X$: $X_1, X_2, \ldots, X_n$

Def: Empirical CDF $F_n(x)$ based on the $X_i$'s

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathscr{I}_{[0,x)}(x_i)$$

where $\mathscr{I}_{[0,x)}$ is the characteristic function of the interval $[0, x)$

# The Kolmogorov-Smirnov Test

The K-S Test is based on $F(x) - F_n(x)$

$$K_n^+ = \sqrt{n} \max_{-\infty < x < +\infty} (F_n(x) - F(x))$$

maximum deviation when $F_n$ is greater than F.

$$K_n^- = \sqrt{n} \max_{-\infty < x < +\infty} (F(x) - F_n(x))$$

maximum deviation when $F_n$ is less than F.

$K_n = \max(K_n^+, K_n^-)$, table like Chi-Square to find the percentile, but unlike $\chi^2$, the table fits any size of n.

# The Kolmogorov - Smirnov Test

Simple procedure to obtain $K_n^+$, $K_n^-$, to test hypothesis that $X_i \sim F$

1. Obtain observations $X_1, X_2, \ldots, X_n$.
2. Rearrange (sort) into ascending order (with renumbering).

$$X_1 \leq X_2 \leq \ldots \leq X_n$$

3. Calculate $K_n^+$, $K_n^-$

$$K_n^+ = \sqrt{n} \max_{1 \leq j \leq n} \left( \frac{j}{n} - F(X_j) \right), \; K_n^- = \sqrt{n} \max_{1 \leq j \leq n} \left( F(X_j) - \frac{j-1}{n} \right)$$

# The Kolmogorov-Smirnov Test

Dilemma: We need a large $n$ to differentiate $F_n$ and $F$.
Large $n$ will average out local random behavior.

Compromise: Consider a moderate size for $n$, say 1000.
Make a fairly large number of $K^+_{1000}$ on different parts of the random sequence $K^+_{1000}(1), K^+_{1000}(2), \ldots, K^+_{1000}(r)$.
Apply another KS Test. The distribution of $K^+_n$ is approximated.

$$F_\infty(x) = 1 - e^{-2x^2}$$

Significance: Detects both local and global random behavior.

# Empirical Tests

Empirical Tests: 10 tests

Test of real number sequence

$$< \mathcal{U}_n > = \mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2 \ldots$$

Test of integer number sequence

$$< \mathcal{Y}_n > = \mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2 \ldots$$

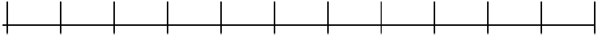
$$\mathcal{Y}_n = \lfloor d\mathcal{U}_n \rfloor$$

$$\mathcal{Y}_n : integers[0, d-1]$$

# A. Equidistribution Test (Frequency Test)

Two ways:

1. Use $\chi^2$ test



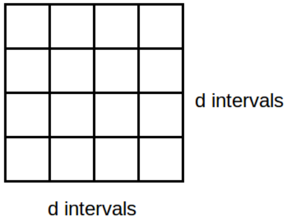
d intervals

Count the number of sequence $<\mathcal{Y}_n>= \mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \ldots$ falling into each interval
k=d
$p_s = \frac{1}{d}$

2. Use KS Test
Test $<\mathcal{U}_n>= \mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$
F(x) = x for $0 \leq x \leq 1$

# B. Serial Test

- Pairs of successive numbers to be uniformly distributed.
- $d^2$ intervals are used.




$$k = d^2,\ p_s = 1/d^2$$

- Serial Test can be regarded as 2-D frequency test.
- Can be generalized to triples, quadruples, . . .

# C. Gap Test

Examine length of "gaps" between occurrences of $\mathcal{U}_j \in \mathcal{I} = (\alpha, \beta)$, where $0 \leq \alpha < \beta \leq 1$, and $p = \beta - \alpha$. A gap is the length $r$ where Length of $\mathcal{U}_j, \mathcal{U}_{j+1}, \ldots, \mathcal{U}_{j+r}$ have $\mathcal{U}_j, \mathcal{U}_{j+r} \in \mathcal{I}$ and all the other are not. Algorithm:

1. Initialize: j$\leftarrow$ -1, s $\leftarrow$ 0
2. r $\leftarrow$ 0
3. if $(\alpha \leq \mathcal{U}_j \leq \beta)$, j $\leftarrow$ j+1
   else goto 5.
4. r $\leftarrow$ r+1, goto 3.
5. record gap length.
   if r $\geq$ t, $\mathcal{COUNT}$[t] $\leftarrow$ $\mathcal{COUNT}$[t]+1
   else $\mathcal{COUNT}$[r] $\leftarrow$ $\mathcal{COUNT}$[r]+1
6. Repeat until $n$ gaps are found.

# C. Gap Test

$\mathcal{COUNT}$[0], $\mathcal{COUNT}$[1], ..., $\mathcal{COUNT}$[t] should have the following probability:

- $p_0 = p, p_1 = p(1-p), p_2 = p(1-p)^2, \ldots, p_{t-1} = p(1-p)^{t-1}, p_t = p(1-p)^t$

Now, we can apply the $\chi^2$ test.
Special cases:

- $(\alpha,\beta) = (0,\frac{1}{2}) \leftarrow$ runs above the mean
- $(\alpha,\beta) = (\frac{1}{2},1) \leftarrow$ runs below the mean

# D. Poker Test

Consider 5 successive integers $(\mathcal{Y}_{sj}, \mathcal{Y}_{sj+1}, \mathcal{Y}_{sj+2}, \mathcal{Y}_{sj+3}, \mathcal{Y}_{sj+4})$

| Pattern | Example | Pattern | Example |
|---|---|---|---|
| All different | abcde | Full house | aaabb |
| One Pair | aabcd | Four of a kind | aaaab |
| Two Pairs | aabbc | Five of a kind | aaaaa |
| Three of a kind | aaabc | | |

Simplify:

| | |
|---|---|
| 5 different | all different |
| 4 different | one pair |
| 3 different | two pairs or three of a kind |
| 2 different | full house or four of a kind |
| 5 same numbers | five of a kind |

# D. Poker Test

Generalized:
*n* groups of *k* successive numbers (*k - tuples*) with *r* different values.

$$pr = \frac{d(d-1)\ldots(d-r+1)}{d^k} \left\{ \begin{matrix} k \\ r \end{matrix} \right\}$$

$$d = number \;\; of \;\; categories$$

Then, the $\chi^2$ test can be applied.

# Stirling Numbers of the Second Kind

- Notation: $S(n,k)$ or $\left\{ {n \atop k} \right\}$
- Definition: counts the number of ways to partition a set of $n$ labelled objects into $k$ nonempty unlabelled subsets **or**
- Also counts the number of different equivalence relations with precisely $k$ equivalence classes that can be defined on an $n$ element set
- Obviously, $\left\{ {n \atop n} \right\} = 1$ and for $n \geq 1$, $\left\{ {n \atop 1} \right\} = 1$: the only way to partition an "n"-element set into "n" parts is to put each element of the set into its own part, and the only way to partition a nonempty set into one part is to put all of the elements in the same part.
- They can be calculated using the following explicit formula:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

# E.Coupon Collector's Test

In the sequence $\mathcal{Y}_0, \mathcal{Y}_1, \ldots$, the lengths of the segments $\mathcal{Y}_{j+1}, \mathcal{Y}_{j+2}, \ldots, \mathcal{Y}_{j+r}$ are collected to get a complete set of integers from 0 to d-1.
Algorithm:

1. Initialize j $\leftarrow$ -1, s $\leftarrow$ 0, $\mathcal{COUNT}$[r] $\leftarrow$ 0 for d $\leq$ r <t.
2. q $\leftarrow$ r $\leftarrow$ 0, $\mathcal{OCCURS}$[k] $\leftarrow$ 0 for 0 $\leq$ k <d.
3. r $\leftarrow$ r+1, j $\leftarrow$ j+1
4. Complete Set? $\mathcal{OCCURS}$[ $\mathcal{Y}_j$ ] $\leftarrow$ 1 and q $\leftarrow$ q+1
   if q=d, a complete set
   q <d, goto 3.
5. Record the length.
   if r $\geq$ t, $\mathcal{COUNT}$[t] $\leftarrow$ $\mathcal{COUNT}$[t]+1
   else $\mathcal{COUNT}$[r] $\leftarrow$ $\mathcal{COUNT}$[r]+1
6. Repeat until n values are found.

# E. Coupon Collector's Test

Chi-Square Test can be applied to $\mathcal{COUNT}$[d], $\mathcal{COUNT}$[d+1], ..., $\mathcal{COUNT}$[t]

$$p_r = \frac{d!}{d^r}\left\{ {r-1 \atop d-1} \right\}, d \leq r < t$$

$$p_t = 1 - \frac{d!}{d^{t-1}}\left\{ {t-1 \atop d} \right\}$$

# F. Permutation Test

A t-tuple $(\mathcal{U}_{jt}, \mathcal{U}_{jt+1}, \ldots, \mathcal{U}_{jt+t-1})$ can have t! possible relative orderings.

For Example: t=3
There should be 3! = 6 categories

| | | |
|---|---|---|
| 1 <2 <3 | 2 <1 <3 | 2 <3 <1 |
| 1 <3 <2 | 3 <1 <2 | 3 <2 <1 |

$$k = t! \qquad p_s = \frac{1}{t!}$$

We can apply $\chi^2$ test now.

# G. Run Test

Examine the length of monotone subsequences.
"Runs up": increasing "Runs down": decreasing
For Run *i*, the length of the run is $\mathcal{COUNT}$[i].

$$| \underbrace{1\ \ 2\ \ 9}_{3} | \underbrace{8}_{1} | \underbrace{5}_{1} | \underbrace{3\ \ 6\ \ 7}_{3} | \underbrace{0\ \ 4}_{2} |$$

Note: $\chi^2$ test cannot be directly applied because of lack of independence (each segment depends on previous segment).

Then, we need to calculate

$$\chi^2 = \frac{1}{n} \sum_{1 \le i,j \le 6} \left(\mathcal{COUNT}[i] - nb_i\right)\left(\mathcal{COUNT}[j] - nb_j\right) a_{ij}$$

# G. Runs Test

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} = \begin{bmatrix} 4529.4 & 9044.9 & 13568 & 18091 & 22615 & 27892 \\ 9044.9 & 18097 & 27139 & 36187 & 45234 & 55789 \\ 13568 & 27139 & 40721 & 54281 & 67852 & 83685 \\ 10891 & 36187 & 54281 & 72414 & 90470 & 111580 \\ 22615 & 45234 & 67852 & 90470 & 113262 & 139476 \\ 27892 & 55789 & 83685 & 111580 & 139476 & 172860 \end{bmatrix}$$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{5}{24} & \frac{11}{120} & \frac{19}{720} & \frac{29}{5040} & \frac{1}{890} \end{pmatrix}$$

Then, $\chi^2$ should have the same $\chi^2$ distribution with degree 6.

# H. Maximum-of-*t* Test

Examine the maximum value of *t* uniform random variables.

Let $\chi^2{}_j = \max(\mathcal{U}_{tj}, \mathcal{U}_{tj+1}, \ldots, \mathcal{U}_{tj+t-1})$.

The distribution is $F(x) = X^t$

Then, we can apply the Kolmogorov - Smirnov Test here.

# I. Collision Test

Suppose we have m urns and n balls, m «n.
Most of the balls will fall in an empty urn.
If a ball falls in an urn that already has a ball, we call it a "collision".

A generator passes the collision test only if it doesn't induce too many or too few collisions.

Probability of $c$ collisions occurring:

$$\frac{m(m-1)\dots(m-n+c+1)}{m^n}\left\{ {n \atop n-c} \right\}$$

# J. Serial Correlation Test

Consider the observations $(\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{n-1})$ and $(\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}, \mathcal{U}_0)$
Test the correlation between these two tuples.
We compute:

$$C = \frac{n(\mathcal{U}_0\mathcal{U}_1 + \mathcal{U}_1\mathcal{U}_2 + \ldots + \mathcal{U}_{n-2}\mathcal{U}_{n-1} + \mathcal{U}_{n-1}\mathcal{U}_0) - (\mathcal{U}_0 + \mathcal{U}_1 + \ldots + \mathcal{U}_{n-1})^2}{n(\mathcal{U}_0^2 + \mathcal{U}_1^2 + \ldots + \mathcal{U}_{n-1}^2) - (\mathcal{U}_0 + \mathcal{U}_1 + \ldots + \mathcal{U}_{n-1})^2}$$

A “good” C should be between $\mu_n$ - 2$\delta_n$ and $\mu_n$ + 2$\delta_n$.

$$\mu_n = \frac{-1}{n-1} \quad , \quad \delta_n = \frac{1}{n-1}\sqrt{\frac{n(n-3)}{n+1}} \quad , \quad n > 2$$

# The Spectral Test

Idea underlying the test: *Congruential Generators generate random numbers in grids!*

In t-dimensional space, $\{(\mathcal{U}_n, \mathcal{U}_{n+1}, \ldots, \mathcal{U}_{n+t-1})\}$

Compute the distance between lines (2D), planes (3D), parallel hyperplanes (>3D).

| | |
|---|---|
| $1/\nu_2$: | Maximum distance between lines.<br>Two dimensional accuracy. |
| $1/\nu_3$: | Maximum distance between planes.<br>Three dimensional accuracy. |
| $1/\nu_t$: | Maximum distance between hyperplanes.<br>$t$ - dimensional accuracy. |

# The Spectral Test

Differentiate between truly random sequences and periodic sequences.

| | |
|---|---|
| Truly random sequences: | accuracy remains same in all dimensions |
| Periodic sequences: | accuracy decreases as t increases |

Spectral Test is by far the most powerful test.

- All “good” generators pass it.
- All known “bad” generators fail it.

# Summary

1. Basic idea of empirical tests:
   The combination of random numbers is expected to conform to a specific distribution.
   1.1 Build the combination.
   1.2 Use $\chi^2$ or KS test to test the deviation from the expected distribution.
2. We can perform an infinite number of tests.
3. We might be able to construct a test to “kill” a specific generator.

# Other resources for RNG Testing

1. FFT, Metropolis, Wolfgang Tests (spectrum).
2. Diehard (`http://www.stat.fsu.edu/pub/diehard`)
3. SPRNG (implements most of the empirical tests and spectrum tests). `http://sprng.cs.fsu.edu`
4. TestU01