Monte Carlo Methods: Early History and The Basics

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Outline of the Talk

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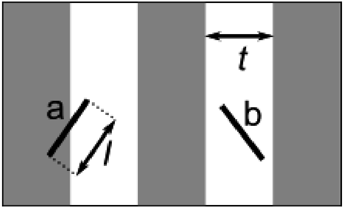
  4. 1812: Laplace, *Théorie Analytique des Probabilités*

Early History of Monte Carlo: Before Los Alamos

- Buffon Needle Problem: Early Monte Carlo (experimental mathematics)



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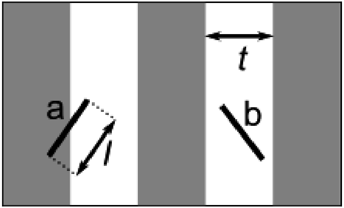

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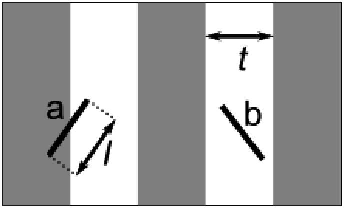

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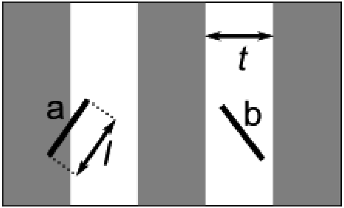

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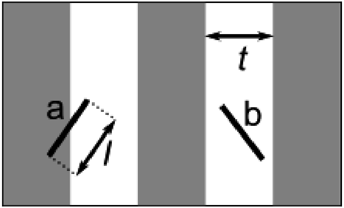

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- In the 1930’s, Fermi used sampling methods to estimate quantities involved in controlled fission

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- The Name: Ulam's uncle would borrow money from the family by saying that “I just have to go to Monte Carlo”

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  2. Geometry is problematic for deterministic methods but not for MC

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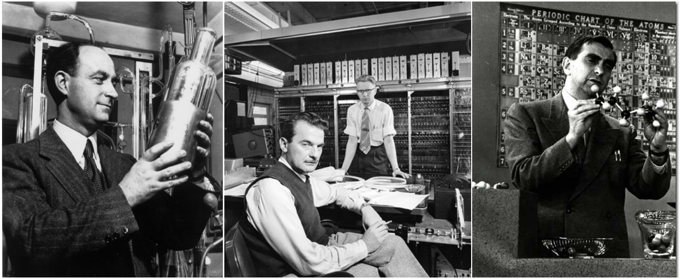

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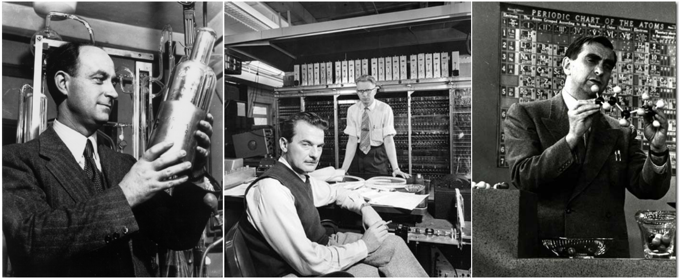

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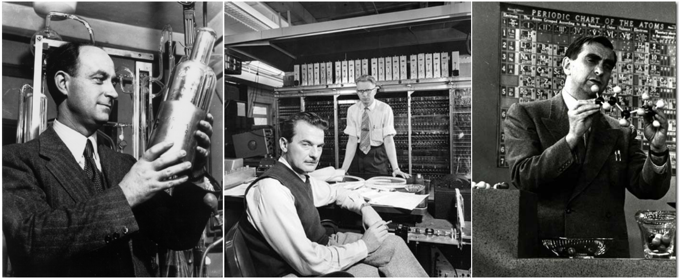

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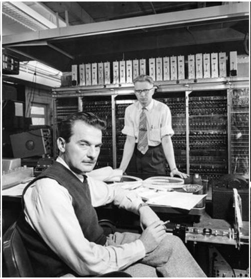

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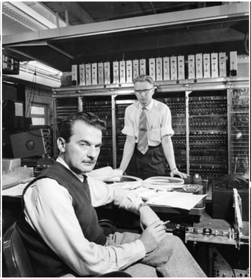
  3. Edward Teller: more interested in the “super”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 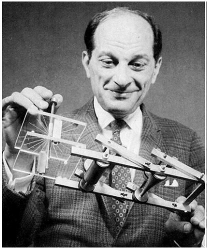 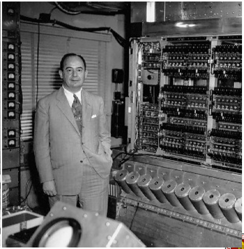

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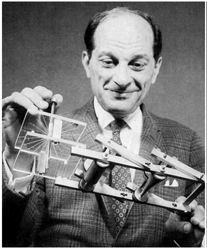

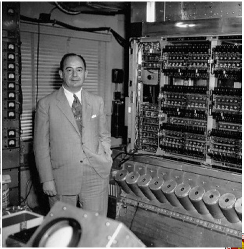

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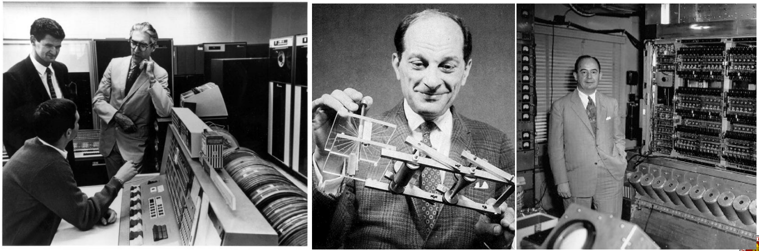

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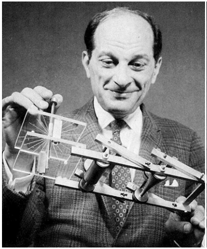

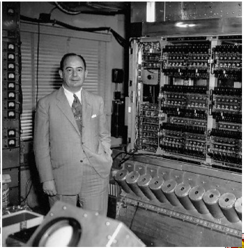

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  4. Continued development and acquisition of digital computers by Metropolis including the MANIAC

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- Parallelism is achievable with the Fermiac

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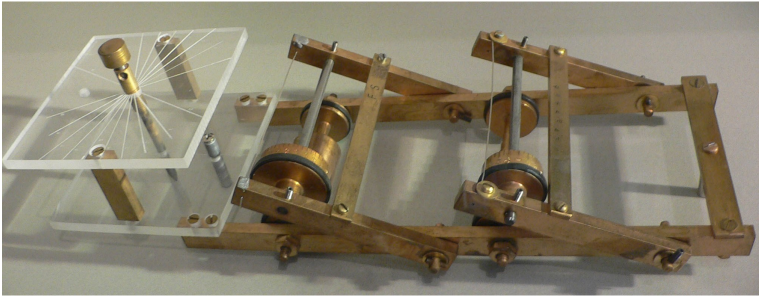

Figure: Enrico Fermi's Fermiac at the Bradbury Museum in Los Alamos

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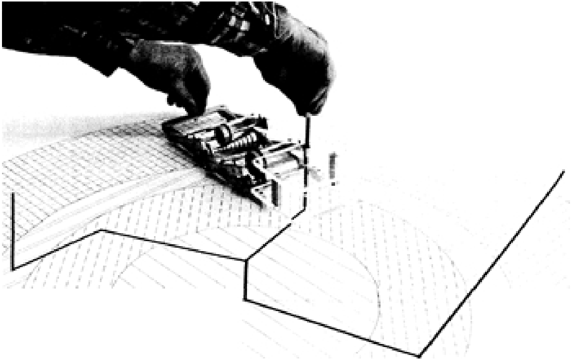

Figure: The Fermiac in Action

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- Remained in continuous operation at the Army BRL until 1955

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- Metropolis would go to BRL to work on the “Los Alamos” problem on the ENIAC

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Figure: The ENIAC at the University of Pennsylvania

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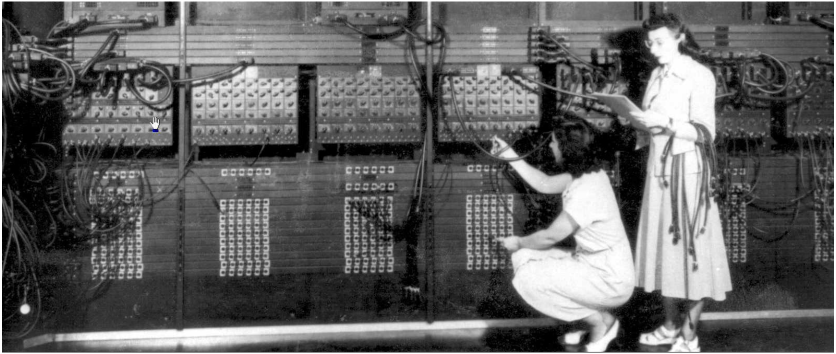

Figure: Programming the ENIAC

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  1. 1952, Los Angeles: RAND Corp., National Bureau of Standards (NBS now NIST), Oak Ridge
  2. 1954, Gainesville, FL: University of Florida Statistical Lab

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   - The random process/variable: $x_i \sim U[0, 1]$ i.i.d.
   - The score: $f(x_i)$
   - One averages and uses a confidence interval for an error bound

$$\bar{I} = \frac{1}{N}\sum_{i=1}^{N} f(x_i), \quad var(I) = \frac{1}{N-1}\sum_{i=1}^{N}(f(x_i)-\bar{I})^2 = \frac{1}{N-1}\left[\sum_{i=1}^{N} f(x_i)^2 - N\bar{I}^2\right],$$

$$var(\bar{I}) = \frac{var(I)}{N}, \quad I \in \bar{I} \pm k \times \sqrt{var(\bar{I})}$$

Other Early Monte Carlo Applications

- Numerical linear algebra based on sums: $S = \sum_{i=1}^{N} a_i$

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- Formally, the solution is $x = (I - H)^{-1} b$

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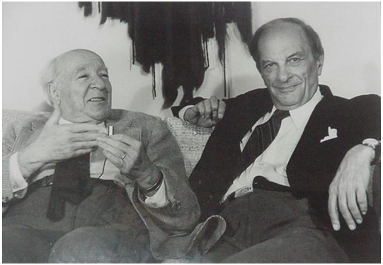
- Methods for partial differential and integral equations



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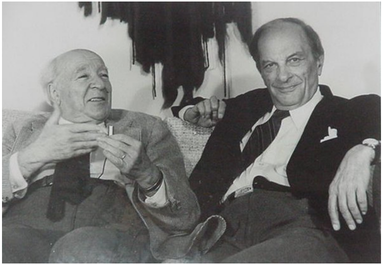

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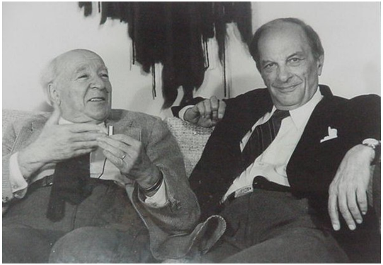

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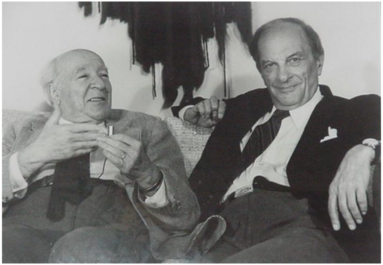

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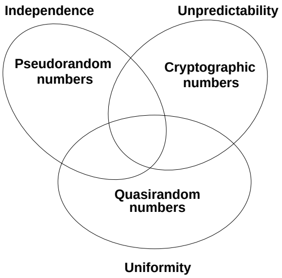

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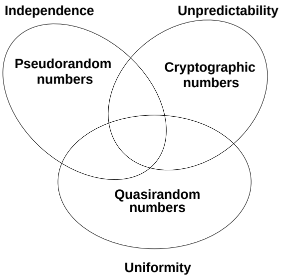

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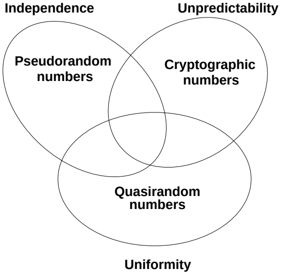

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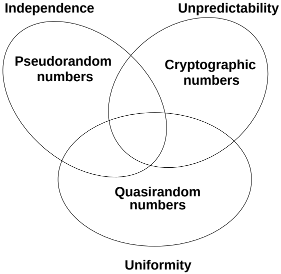

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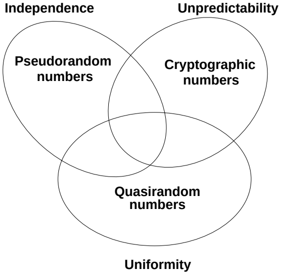

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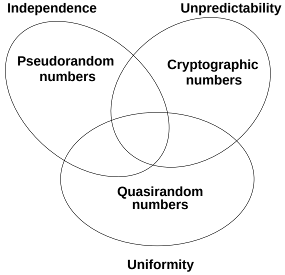

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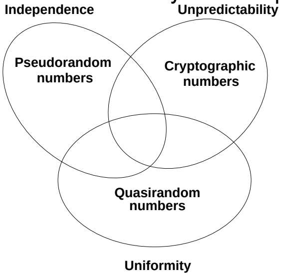

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References

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Questions?