An Introduction to Brownian Motion, Wiener Measure, and Partial Differential Equations

Prof. Michael Mascagni

Applied and Computational Mathematics Division, Information Technology Laboratory
National Institute of Standards and Technology, Gaithersburg, MD 20899-8910 **USA**
AND
Department of Computer Science
Department of Mathematics
Department of Scientific Computing
Graduate Program in Molecular Biophysics
Florida State University, Tallahassee, FL 32306 **USA**

E-mail: mascagni@fsu.edu or mascagni@math.ethz.ch
or mascagni@nist.gov
URL: http://www.cs.fsu.edu/∼mascagni
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Outline of the Lectures

Introduction to Brownian Motion

- Let $\Omega = \{\beta \in C[0,1]; \beta(0) = 0\} \overset{\text{def}}{=} C_0[0,1]$, be an infinitely dimensional space we consider for placing a probability measure
- Consider $(\Omega, \mathcal{B}, P)$, where $\mathcal{B}$ is the set of measurable subsets (a σ-*algebra*) and P is the probability measure on Ω
- We would like to answer questions like $P\left[\int_0^1 \beta^2(s)ds \leq \alpha\right]$?
- We now construct Brownian motion (BM) via some limit ideas
- Central Limit Theorem (CLT): let $X_1, X_2, \ldots$ be independent, identically distributed(i.i.d.) with $E[X_i] = 0,\ Var[X_i] = 1$ and define $S_n = \sum_{i=1}^n X_i$
  1. Note if $X_1^*, X_2^*, \ldots$ are i.i.d. with $E[X_i^*] = \mu,\ Var[X_i^*] = \sigma^2 < \infty$, then $X_i = \frac{X_i^* - \mu}{\sigma}$ has $E[X_i] = 0,\ Var[X_i] = 1$
  2. Then $\frac{S_n}{\sqrt{n}}$ converges in distribution to $N(0,1)$ as $n \to \infty$

Introduction to Brownian Motion

- Let $X_1, X_2, \ldots$ be as before, then it follows from the CLT that

$$\lim_{n\to\infty} P\left[\frac{S_n}{\sqrt{n}} \leq \alpha\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}}\, du.$$

- Erdös and Kac proved (we will find the $\sigma_i(\cdot)$'s):
  1. $\lim_{n\to\infty} P\left[\max\left(\frac{S_1}{\sqrt{n}}, \frac{S_2}{\sqrt{n}}, \ldots, \frac{S_n}{\sqrt{n}}\right) \leq \alpha\right] = \sigma_1(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} e^{-\frac{u^2}{2}}\, du$
  2. $\lim_{n\to\infty} P\left[\frac{S_1^2 + S_2^2 + \cdots + S_n^2}{n^2} \leq \alpha\right] = \sigma_2(\alpha)$
  3. $\lim_{n\to\infty} P\left[\frac{S_1 + S_2 + \cdots + S_n}{n^{3/2}} \leq \alpha\right] = \sigma_3(\alpha)$
- Let $N_n = \#\{S_1, \ldots, S_n | S_i > 0\}$, then

$$\lim_{n\to\infty} P\left[\frac{N_n}{n} \leq \alpha\right] = \begin{cases} 0, & \text{if } \alpha \leq 0 \\ \frac{2}{\pi} \arcsin\sqrt{\alpha}, & \text{if } 0 \leq \alpha \leq 1 \\ 1, & \text{if } \alpha \geq 1 \end{cases}$$

Definitions

- $X_1, X_2, \ldots$ are as above, and $\forall n \in \mathbb{N}$ and $t \in [0,1]$ define

$$\chi^{(n)}(t) = \begin{cases} \frac{S_1}{\sqrt{n}}, & t = 0 \\ \frac{S_i}{\sqrt{n}}, & \frac{i-1}{n} < t \leq \frac{i}{n}, \quad i = 1, 2, \ldots, n \end{cases}$$

- Let $\mathcal{R}$ denote the space of Riemann integrable functions on $[0,1]$.
- Theorem: $F : \mathcal{R} \to \mathbb{R}$ and with some weak hypotheses, then

$$\lim_{n\to\infty} P\left[F\left(\chi^{(n)}(\cdot)\right) \leq \alpha\right] = P_W\left[F\left(\beta\right) \leq \alpha\right],$$

where P_W denotes the probability called "Wiener measure," and this result is called Donsker's Invariance Principal

Examples of Donsker's Invariance Principal

1. $F[\beta] = \int_0^1 \beta^2(s)\, ds$, then by the theorem

$$\lim_{n\to\infty} P\left[\sum_{i=1}^{n} \frac{S_i^2}{n^2} \leq \alpha\right] = P_W\left[\int_0^1 \beta^2(s) ds \leq \alpha\right]$$

2. $F[\beta] = \beta(1)$, then

$$\lim_{n\to\infty} P\left[\frac{S_n}{\sqrt{n}} \leq \alpha\right] = P_W\left[\beta(1) \leq \alpha\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}}\, du$$

3. $F[\beta] = \int_0^1 \frac{1+\mathrm{sgn}\beta(s)}{2}\, ds$, where $\mathrm{sgn}(x) = \begin{cases} 1, & : \quad x > 0 \\ -1, & : \quad x \leq 0 \end{cases}$ Then

$$\lim_{n\to\infty} P\left[\frac{N_n}{n} \leq \alpha\right] = P_W\left[\int_0^1 \frac{1+\mathrm{sgn}\beta(s)}{2}\, ds \leq \alpha\right]$$

Defining Wiener Measure Using Cylinder Sets

- For any integer n, any choice of $0 < \tau_1 < \cdots < \tau_n \leq 1$, and any Lebesgue measurable ($\mathcal{L}$-mb) set, $E \in \mathbb{R}^n$ define the "interval"

$$I = I(n; \tau_1; \ldots; \tau_n; E) := \{\beta(\cdot) \in C_0[0,1]; (\beta(\tau_1), \ldots, \beta(\tau_n)) \in E\}$$

- Let $\mathcal{A}$ be the class of intervals containing all the I for all $n, \tau_1, \ldots, \tau_n$ and all $\mathcal{L}-$mb sets $E \in \mathbb{R}^n$, then $\mathcal{A}$ is an algebra of sets in $C_0[0,1]$
- The I's are the cylinder sets upon which we will define Wiener measure, and then standard measure theoretic ideas to extend to all measurable subsets of the infinite dimensional space, $C_0[0,1]$

Defining Wiener Measure Using Cylinder Sets

- Given I, we define its measure as

$$\mu(I) = \frac{1}{\sqrt{(2\pi)^n \tau_1(\tau_2 - \tau_1)\cdots(\tau_n - \tau_{n-1})}}$$
$$\int \cdots \int_E e^{-\frac{u_1^2}{2\tau_1} - \frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)} - \cdots - \frac{(u_n-u_{n-1})^2}{2(\tau_n-\tau_{n-1})}} du_1 \cdots du_n.$$

- Let $\mathcal{B}$ be the smallest $\sigma-$algebra generated by $\mathcal{A}$, this is the class of Wiener measurable (W-mb) sets in $C_0[0,1]$
- This extension of Wiener measure, also creates a probability measure on $C_0[0,1]$, and expectation w.r.t. Wiener measure will be referred to as a
  1. Wiener integral or Wiener integration
  2. Brownian motion expectation

Examples

- Let $A \in \mathbb{R}^{n\times n}$ with $A_{ij} = \min(\tau_i, \tau_j)$, i.e for the case $n = 3$, $\quad \tau_1 < \tau_2 < \tau_3$ we have

$$A = \begin{pmatrix} \tau_1 & \tau_1 & \tau_1 \\ \tau_1 & \tau_2 & \tau_2 \\ \tau_1 & \tau_2 & \tau_3 \end{pmatrix}$$

and in general we can write $U = (u_1, \ldots, u_n)^\top$ and

$$\mu(I) = \frac{1}{\sqrt{(2\pi)^n \det A}} \int \cdots \int_E e^{-U^\top A^{-1} U}\, du_1 \ldots du_n$$

- Let $\beta(\cdot)$ be a BM, and $0 < \tau_1 < \tau_2 < 1$, then

$$P[a_1 \leq \beta(\tau_1) \leq b_1] = \frac{1}{2\pi\tau_1} \int_{a_1}^{b_1} e^{-\frac{u^2}{2\tau_1}}\, du \text{ and}$$

$$P[a_1 \leq \beta(\tau_1) \leq b_1 \cap a_2 \leq \beta(\tau_2) \leq b_2]$$

$$= \frac{1}{\sqrt{(2\pi)^2 \tau_1 (\tau_2 - \tau_1)}} \int_{a_2}^{b_2} \int_{a_1}^{b_1} e^{-\frac{u^2}{2\tau_1} - \frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)}}\, du_1\, du_2$$

Useful Properties of Brownian Motion

- Theorem: Let $I = \bigcup_{j=1}^{\infty} I_j$ where $I_j \cap I_k = \emptyset \, \forall i \neq k$ and $I, I_1, I_2, \dots \in \mathcal{A}$, then $\mu(I) = \sum_{j=1}^{\infty} \mu(I_j)$
- we will see that the BM, $\beta(t)$, satisfies:
  1. Almost every (AE) path is non-differentiable at every point
  2. AE path satisfies a Hölder condition of order $\alpha < \frac{1}{2}$, i.e.
     $$|\beta(s) - \beta(t)| \leq L|s-t|^{\alpha}$$
  3. $E[\beta(t)] = 0$
  4. $E[\beta^2(t)] = t$, and so $\beta(t) \sim N(0, t)$
  5. $\beta(0) = 0, \ \beta(t) - \beta(s) \sim N(0, t - s)$
  6. $E[\beta(t)\beta(s)] = \min(s, t)$

Useful Properties of Brownian Motion

- Let $E \in \mathbb{R}^n\,(\mathcal{L}-mb)$, $0<\tau_1<\cdots<\tau_n<1$, $I = I(n;\tau_1;\ldots;\tau_n;E)$, then

$$\mu(I) = \int \cdots \int_E p(\tau_1, 0, u_1)p(\tau_2-\tau_1, u_1, u_2)\cdots \\ p(\tau_N-\tau_{n-1}, u_n, u_{n-1})\,du_1\cdots du_n$$

  where $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}$
- Note that $p(t,x,y) = \psi(t,x,y)$, the fundamental solution for the initial value problem for the heat/diffusion equation

$$\psi_t = \frac{1}{2}\psi_{yy}, \quad \psi(0,x,y) = \delta(y-x)$$

- μ is finitely additive since integrals are additive set functions

Useful Properties of Brownian Motion

- Theorem 1: Let $a > 0$, $0 < \gamma < \frac{1}{2}$ and define

$$A_{a,\gamma} = \{\beta \in C_0[0,1]; |\beta(\tau_2) - \beta(\tau_1)| \leq a|\tau_2 - \tau_1|^{\gamma}\, \forall \tau_1, \tau_2 \in [0,1]\}$$

  For any interval $I \subset C_0[0,1]$ s.t. $I \cap A_{a,\gamma} = \emptyset$ there is a K indepedent of a for which

$$m(I) < Ka^{-\frac{4}{1-2\gamma}}$$

- Remark: $A_{a,\gamma}$ is a compact set in $C_0[0,1]$ and eventually one can prove that AE $\beta \in C_0[0,1]$ satisfy some Hölder condition
- Theorem 2: μ is countably additive on $\mathcal{A}$, i.e. if $I_n \in \mathcal{A}$, $n \in \mathbb{N}$ disjoint ($I_j \cap I_k = \emptyset$, $j \neq k$) then

$$I = \bigcup_{n=1}^{\infty} I_n \in \mathcal{A} \Rightarrow \mu(I) = \sum_{n=1}^{\infty} \mu(I_n)$$

Useful Properties of Brownian Motion

- Suppose $F : C_0[0,1] \to \mathbb{R}$ is a measurable functional, i.e. $\{\beta \in C_0[0,1]; F[\beta] \leq \alpha\}$ is measurable $\forall \alpha$
- We can consider

$$E[F] = E_W[F[\beta(\cdot)]] = \int F[\beta(\cdot)]\delta_W, \text{ a Wiener integral}$$

- Consider $C_x[0,t] = \{f \in C[0,t]; f(0) = x\}$, then

$$P\left[\beta(0) = x, \beta(t) \in A\right] = \frac{1}{\sqrt{2\pi t}} \int_A e^{-\frac{(y-x)^2}{2t}}\, dy$$

- Furthermore

$$E\left[\beta(\tau)\right] = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2\tau}}\, du = 0, \ \forall \tau > 0$$

$$E\left[g\left(\beta(\tau_1), \dots, \beta(\tau_n)\right)\right] = \frac{1}{\sqrt{(2\pi)^n \tau_1(\tau_2 - \tau_1)\cdots(\tau_n - \tau_{n-1})}} \times$$

$$\int \cdots \int g(u_1, \dots, u_n) e^{-\frac{u_1^2}{2\tau_1} - \frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)} - \cdots - \frac{(u_n-u_{n-1})^2}{2(\tau_n-\tau_{n-1})}}\, du_1 \cdots du_n$$

Useful Properties of Brownian Motion

- Let us now consider, without proof, a large deviation result for BM:
- Theorem (The Law of the Iterated Logarithm for BM): Let $\beta(s) \in C_0[0, \infty)$ be ordinary Brownian Motion, then

(1)
$$P\left(\limsup_{t\to\infty} \frac{\beta(t)}{\sqrt{2t \ln \ln t}} = 1\right) = 1$$

(2)
$$P\left(\liminf_{t\to\infty} \frac{\beta(t)}{\sqrt{2t \ln \ln t}} = -1\right) = 1$$

Dirac Delta Function

- Let g be Borel measurable (B-mb), then

$$E[g(\beta(\tau))] = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} g(u) e^{-\frac{u^2}{2\tau}}\, du$$

- Let $g(u) = \delta(u - x)$, using the Dirac delta function, then

$$E[\delta(\beta(t) - x)] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \delta(u - x) e^{-\frac{u^2}{2t}}\, du = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

thus $u(x,t) = E[\delta(\beta(t) - x)] = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ is the fundamental solution of the heat equation

$$u_t = \frac{1}{2} u_{xx}, \;\; u(x, 0) = \delta(x)$$

The Feynman-Kac Formula

- Consider now $V(x) \geq 0$ continuous and consider the equation

$$u_t = \frac{1}{2}u_{xx} - V(x)u, \;\; u(x,0) = \delta(x),$$

  then we can write

$$u(x,t) = E\left[e^{-\int_0^t V(\beta(s))\,ds}\delta(\beta(t) - x)\right]$$

  This is the Feynman-Kac formula

- Example:

$$V(x) = \frac{x^2}{2}, \;\; u_t = \frac{1}{2}u_{xx} - \frac{x^2}{2}u, \;\; u(x,0) = \delta(x), \text{ then}$$
$$u(x,t) = E\left[e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds}\delta(\beta(t) - x)\right]$$

The Feynman-Kac Formula

- The following is clearly true:

$$P[\beta(\tau) \leq x] = P\left(\{\beta \in C_0[0,\tau]; \beta(\tau) \in E = (-\infty, x]\}\right) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{x} e^{-\frac{u^2}{2\tau}}\, du, \text{ and similarly}$$

With $0 = \tau_0 \leq \tau_1 \cdots \leq \tau_n$ we have

$$P\left[\beta(\tau_1) \leq x_1, \ldots, \beta(\tau_n) \leq x_n\right] = \frac{(2\pi)^{-n/2}}{\sqrt{(\tau_1 - \tau_0)(\tau_2 - \tau_1) \cdots (\tau_n - \tau_{n-1})}} \times$$

$$\int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} e^{-\frac{u_1^2}{2\tau_1} - \frac{(u_2 - u_1)^2}{2(\tau_2 - \tau_1)} - \cdots - \frac{(u_n - u_{n-1})^2}{2(\tau_n - \tau_{n-1})}}\, du_1 \cdots du_n$$

- Hence with $A_{ij} = \min(\tau_i, \tau_j)$

$$E\left[g\left(\beta(\tau_1), \ldots, \beta(\tau_n)\right)\right] = \frac{1}{\sqrt{(2\pi)^n |A|}} \times$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(u_1, \cdots, u_n) e^{-\frac{1}{2} U^\top A^{-1} U}\, du_1 \cdots du_n$$

Feynman-Kac Formula: Derivation

- Let us consider the Wiener integral below, where expectation is taken over all of $C_0[0, t]$

$$E\left\{e^{-\int_0^t V(\beta(\tau))\,d\tau}\right\}$$

- We will show that this is equal to the solution of the Bloch equation using an elementary proof of Kac
- We assume that $0 \leq V(x) < M$ is bounded from above and non-negative; however, the upper bound will be relaxed
- We know

$$E\left\{e^{-\int_0^t V(\beta(\tau))\,d\tau}\right\} = \sum_{k=0}^{\infty}(-1)^k\left[\int_0^t V(\beta(\tau))\,d\tau\right]^k/k!$$

- Since $V(\cdot)$ is bounded we also have

$$0 < \int_0^t V(\beta(\tau))\,d\tau < Mt$$

- This allows us to use Fubini's theorem as follows

$$E\left\{e^{-\int_0^t V(\beta(\tau))\,d\tau}\right\} = \sum_{k=0}^{\infty}(-1)^k E\left\{\left[\int_0^t V(\beta(\tau))\,d\tau\right]^k\right\}/k!$$

Feynman-Kac Formula: Derivation

- Now let us consider the moments

$$\mu_k(t) = E\left\{\left[\int_0^t V(\beta(\tau))\,d\tau\right]^k\right\}$$

- Consider first $k = 1$

$$E\left\{\int_0^t V(\beta(\tau))\,d\tau\right\} \overset{Fubini}{=} \int_0^t E\left\{V(\beta(\tau))\right\}\,d\tau = \int_0^t \int_{-\infty}^{\infty} V(\xi)\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{\xi^2}{2\tau}}\,d\xi\,d\tau$$

- The case $k = 2$ is a bit more complicated

$$E\left\{\left[\int_0^t V(\beta(\tau))\,d\tau\right]^2\right\} = 2!E\left\{\int_0^t \int_0^{\tau_2} V(\beta(\tau_1))V(\beta(\tau_2))\,d\tau_1\,d\tau_2\right\} \overset{Fubini}{=}$$

$$2!\int_0^t \int_0^{\tau_2} E\left\{V(\beta(\tau_1))V(\beta(\tau_2))\right\}\,d\tau_1\,d\tau_2 =$$

$$2!\int_0^t \int_0^{\tau_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\xi_1)V(\xi_2)\frac{e^{-\frac{\xi_1^2}{2\tau_1}}}{\sqrt{2\pi\tau_1}}\frac{e^{-\frac{(\xi_2-\xi_1)^2}{2(\tau_2-\tau_1)}}}{\sqrt{2\pi(\tau_2-\tau_1)}}\,d\xi_1\,d\xi_2\,d\tau_1\,d\tau_2$$

Feynman-Kac Formula: Derivation

- For general k we will proceed by defining the function $Q_n(x,t)$ as follows
  1. $Q_0(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$
  2. $Q_{n+1}(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\tau - t)}} e^{-\frac{(x-\xi)^2}{2(\tau - t)}} V(\xi) Q_n(\xi,\tau)\, d\xi\, d\tau$
- We have that $\mu_k(t) = k! \int_0^t Q_k(x,t)\, dx$
- By the boundedness of $V(\cdot)$ we also have, by induction, that $0 \leq Q_n(x,t) \leq \frac{(Mt)^n}{n!} Q_0(x,t)$
- Now define $Q(x,t) = \sum_{k=0}^{\infty} (-1)^k Q_k(x,t)$
- This series converges for all x and $t \neq 0$ and $|Q(x,t)| < e^{Mt} Q_0(x,t)$
- One can easily check that the definitions of the $Q_k(x,t)$'s ensures that $Q(x,t)$ satisfies the following integral equation

$$Q(x,t) + \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{(t-\tau)}} e^{-\frac{(x-\xi)^2}{2(t-\tau)}} V(\xi) Q(\xi,\tau)\, d\xi\, d\tau = Q_0(x,t)$$

Feynman-Kac Formula: Derivation

- It also follows that
$$E\left\{e^{-\int_0^t V(\beta(\tau))\,d\tau}\right\} = \int_{-\infty}^{\infty} Q(x,t)\,dx$$
- Recall that his Wiener integral is over all of $C_0[0,t]$, let us restrict this only to $a < \beta(t) < b$, thus
$$E\left\{e^{-\int_0^t V(\beta(\tau))\,d\tau}; a < \beta(t) < b\right\} = \int_a^b Q(x,t)\,dx$$
- This tell us immediately that $Q(x,t) \geq 0$
- Now we will relax the upper bound on $V(\cdot)$ by considering the function
$$V_M(x) = \begin{cases} V(x), & \text{if } V(x) \leq M \\ M, & \text{if } V(x) \geq M \end{cases}$$
and we denote $Q^{(M)}(x,t)$ as the respective "Q" function

Feynman-Kac Formula: Derivation

- By the additivity of Wiener measure we have that

$$\lim_{M\to\infty} E\left\{e^{-\int_0^t V_M(\beta(\tau))\,d\tau}; a < \beta(t) < b\right\} = E\left\{e^{-\int_0^t V(\beta(\tau))\,d\tau}; a < \beta(t) < b\right\}$$

- Furthermore, as $M \to \infty$ the functions $Q^{(M)}(x,t)$ form a decreasing sequence with $\lim_{M\to\infty} Q^{(M)}(x,t) = Q(x,t)$ existing with the resulting limiting function, $Q(x,t)$ satisfying the (Bloch) equation

$$\frac{\partial Q}{\partial t} = \frac{1}{2}\frac{\partial^2 Q}{\partial x^2} - V(x)Q$$

with the initial condition $Q(x,t) \to \delta(x)$ as $t \to 0$

Feynman-Kac Formula: Derivation Variation

- Recall the integral equation solved by $Q(x,t)$

$$Q(x,t) + \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{(t-\tau)}} e^{-\frac{(x-\xi)^2}{2(t-\tau)}} V(\xi) Q(\xi,\tau)\, d\xi\, d\tau = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

- Let us define $\Psi(x) = \int_{-\infty}^{\infty} Q(x,t) e^{-st}\, dt$ with $s > 0$, this is the Laplace transform of $Q(x,t)$
- Now multiply the integral equation by e^{-st} and integrate out t to get the equation satisfied by the Laplace transform of $Q(x,t)$

$$\Psi(x) + \frac{1}{\sqrt{2s}} \int_{-\infty}^{\infty} e^{-\sqrt{2s}|x-\xi|} V(\xi)\Psi(\xi)\, d\xi = \frac{1}{\sqrt{2s}} e^{-\sqrt{2s}|x|}$$

- It is easy to verify that $\Psi(x)$ also satisfies the following differential equation

$$\frac{1}{2}\Psi'' - (s + V(x))\Psi = 0, \text{ with the following conditions}$$

1. $\Psi \to 0$ as $|x| \to \infty$
2. Ψ' is continuous except at $x = 0$
3. $\Psi'(-0) - \Psi'(-0) = 2$

Explicit Representation of Brownian Motion

- Suppose that $F[\beta] = \int_0^t \beta^2(s)\, ds$, then it follows

$$E\left[\int_0^t \beta^2(s)\, ds\right] \overset{Fubini}{=} \int_0^t E\left[\beta^2(s)\right]\, ds = \int_0^t s\, ds = \frac{t^2}{2}$$

- To compute $E\left[e^{\int_0^t \beta(s)\, ds}\right]$, we need to do some classical analysis
- Consider the eigenvalue problem for this integral equation

$$\rho \int_0^t u(s) \min(\tau, s)\, ds = u(\tau)$$

- Find eigenvalues $\rho_0, \rho_1, \ldots$ and corresponding orthonormalized eigenfunctions $u_0(\tau), u_1(\tau), \ldots$ with $\int_0^t u_j(\tau) u_k(\tau)\, d\tau = \delta_{jk}, \forall j, k \geq 0$

Explicit Representation of Brownian Motion

- For $t > \tau$ we have

$$\rho \int_0^\tau s u(s)\, ds + \rho \int_\tau^t \tau u(s)\, ds = u(\tau)$$

$$\xRightarrow{\frac{d}{d\tau}} \rho\tau u(\tau) - \rho\tau u(\tau) + \rho \int_\tau^t u(s)\, ds = u'(\tau)$$

$$\xRightarrow{\frac{d}{d\tau}} -\rho u(\tau) = u''(\tau)$$

Thus $u''(\tau) + \rho u(\tau) = 0$ and with $u(0) = 0,\ u'(t) = 0$ we get

$$\left.\begin{aligned} \rho_k &= (k + \tfrac{1}{2})^2 \tfrac{\pi^2}{t^2} \\ u_k(s) &= \sqrt{\tfrac{2}{t}} \sin\left((k + \tfrac{1}{2})\tfrac{\pi s}{t}\right) \end{aligned}\right\} \quad k = 0, 1, 2, \ldots$$

- By the spectral theorem the integral equation kernel can be represented as:

$$\min(s, \tau) = \sum_{k=0}^{\infty} \frac{u_k(s) u_k(\tau)}{\rho_k}$$

Explicit Representation of Brownian Motion

- Let $\alpha_0(\omega), \alpha_1(\omega), \ldots$ be i.i.d. $N(0,1)$, then we claim that the following is an explicit representation of BM

$$\sum_{k=0}^{\infty} \frac{\alpha_k(\omega) u_k(\tau)}{\sqrt{\rho_k}} = \beta(\tau) \tag{2.1}$$

- This is a Fourier series with random coefficients and we will prove that this converges for AE path ω with the following properties
  1. We use ω to denote an individual sample of i.i.d. $N(0,1)$ $\alpha_i(\omega)$'s
  2. $E[\alpha_i(\omega)] = 0, \forall i \geq 0$
  3. $E[\alpha_i(\omega)\alpha_j(\omega)] = \delta_{ij}, \forall i, j \geq 0$
- This is the simplest version of the Karhunen-Loève expansion of stochastic processes

Explicit Representation of Brownian Motion (Proof)

- We now use the representation (2.1) to compute some expectations w.r.t. the α_i's $\sim N(0,1)$

$$E\left[\sum_{k=0}^{\infty}\frac{\alpha_k(\omega)}{\sqrt{\rho_k}}u_k(\tau)\right] \overset{i.i.d.\ N(0,1)\ \&\ Fubini}{=}$$
$$\sum_{k=0}^{\infty}\frac{E[\alpha_k(\omega)]u_k(\tau)}{\sqrt{\rho_k}} = \sum_{k=0}^{\infty}\frac{0\times u_k(\tau)}{\sqrt{\rho_k}} = 0 = E[\beta(\tau)]$$

- We now use the representation (2.1) to compute some expectations

$$E\left[\sum_{k=0}^{\infty}\frac{\alpha_k(\omega)}{\sqrt{\rho_k}}u_k(\tau)\sum_{l=0}^{\infty}\frac{\alpha_l(\omega)}{\sqrt{\rho_l}}u_l(\tau)\right] \overset{i.i.d. N(0,1)}{=}$$
$$\sum_{k=0}^{\infty}\frac{u_k^2(\tau)}{\rho_k} = \min(\tau,\tau) = \tau = E\left[\beta^2(\tau)\right]$$

Explicit Representation of Brownian Motion (Proof)

- Similarly we compute

$$E\left[\sum_{k=0}^{\infty}\frac{\alpha_k(\omega)}{\sqrt{\rho_k}}u_k(\tau)\sum_{l=0}^{\infty}\frac{\alpha_l(\omega)}{\sqrt{\rho_l}}u_l(s)\right]\overset{i.i.d.N(0,1)}{=}$$

$$\sum_{k=0}^{\infty}\frac{u_k(\tau)u_k(s)}{\rho_k}=\min(\tau,s)=\tau=E\left[\beta(\tau)\beta(s)\right]$$

- We have computed the mean, variance, and correlation of the process defined in (2.1), and it is clear that it is $\sim N(0,\tau)$ and hence Brownian motion, $\beta(\tau)$

An Introduction to the Karhunen-Loève Expansion

- Karhunen-Loève (KL) expansion writes the stochastic processes $Y(\omega, t)$ as a stochastic linear combination of a set of orthonormal, deterministic functions in L^2, $\{e_i(t)\}_{i=0}^{\infty}$

$$Y(\omega, t) = \sum_{i=0}^{\infty} Z_i(\omega) e_i(t)$$

1. Given the covariance function of the random process $Y(\omega, t)$ as $C_{YY}(s, \tau)$ the KL expansion is

$$Y(\omega, t) = \sum_{i=0}^{\infty} \sqrt{\lambda_i} \xi_i(\omega) \phi_i(t)$$

2. Here λ_i and $\phi_i(t)$ are the eigenvalues and L^2-orthonormal eigenfunctions of the covariance function and $\xi_i(\omega)\phi_i(t)$ are i.i.d. random variables whose distribution depends on $Y(\omega, t)$, i.e. $Z_i(\omega) = \sqrt{\lambda_i}\xi_i(\omega)$, and $e_i(t) = \phi_i(t)$
3. It can be shown that such an expansion converges to the stochastic process in L^2 (in distribution)

An Introduction to the Karhunen-Loève Expansion

4. By the spectral theorem, we can expand the covariance, thought of as an integral equation kernel, as follows

$$C_{YY}(s,\tau) = \sum_{i=0}^{\infty} \lambda_i \phi_i(s)\phi_i(\tau)$$

5. Here λ_i and $\phi_i(t)$ are the eigenvalues and eigenfunctions of the following integral equation

$$\int_0^{\infty} C_{YY}(s,\tau)\phi_j(\tau)\,d\tau = \lambda_j \phi_j(s)$$

- For ordinary BM, $Y(\omega,t) = \beta(t)$, we have from above

1. $C_{YY}(s,\tau) = C_{\beta\beta}(s,\tau) = \min(s,\tau)$
2. $\lambda_j = \frac{1}{\rho_j}$, where $\rho_j = (j+\frac{1}{2})^2 \frac{\pi^2}{s^2}$
3. $\phi_j(t) = u_j(t) = \sqrt{\frac{2}{s}} \sin((j+\frac{1}{2})\frac{\pi t}{s})$
4. $\xi_j(\omega) = \alpha_j(\omega) \sim N(0,1)$
5. $Y(\omega,t) = \sum_{j=0}^{\infty} \frac{\alpha_j(\omega)u_j(t)}{\sqrt{\rho_j}} = \beta(t)$

Explicit Computation of Wiener Integrals

- We are now in position to compute

$$E\left[e^{\int_0^t \beta(s)\,ds}\right] = E\left[e^{\int_0^t \sum_{k=0}^{\infty} \frac{\alpha_k u_k(s)}{\sqrt{\rho_k}}\,ds}\right] =$$

$$E\left[e^{\sum_{k=0}^{\infty} \int_0^t \frac{\alpha_k}{\sqrt{\rho_k}} u_k(s)\,ds}\right] \overset{indep.}{=} \prod_{k=0}^{\infty} E\left[e^{\frac{\alpha_k}{\sqrt{\rho_k}} \int_0^t u_k(s)\,ds}\right] =$$

$$\prod_{k=0}^{\infty} e^{\frac{1}{2\rho_k}\left(\int_0^t u_k(s)\,ds\right)^2} = e^{\frac{1}{2} \int_0^t \int_0^t \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k}\,ds\,d\tau} =$$

$$e^{\frac{1}{2} \int_0^t \int_0^t \min(s,\tau)\,ds\,d\tau} = e^{\frac{1}{2} \int_0^t [(\frac{\tau^2}{2} + (\tau(t-\tau))]\,d\tau} = e^{\frac{t^3}{6}}$$

- We have used the following results
  1. $E[e^{\alpha u}] = e^{\frac{u^2}{2}}$, with $\alpha \sim N(0,1)$ via moment generating function
  2. $\int_0^t \min(s,\tau)\,ds = \int_0^{\tau} s\,ds + \int_{\tau}^t \tau\,ds = \frac{\tau^2}{2} + (\tau(t-\tau))$

Explicit Computation of Wiener Integrals

- Moreover

$$E\left[e^{-\frac{\lambda^2}{2}\int_0^t \beta^2(s)\,ds}\right] = E\left[e^{-\frac{\lambda^2}{2}\sum_{k=0}^{\infty}\frac{\alpha_k^2}{\rho_k}}\right]$$

$$\overset{indep.}{=} \prod_{k=0}^{\infty} E\left[e^{-\frac{\lambda^2}{2}\frac{\alpha_k^2}{\rho_k}}\right] = \prod_{k=0}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{2}\frac{\alpha^2}{\rho_k}} e^{-\frac{\alpha^2}{2}}\,d\alpha$$

$$= \prod_{k=0}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{2}\left(1+\frac{\lambda^2}{\rho_k}\right)}\,d\alpha$$

$$= \prod_{k=0}^{\infty}\frac{1}{\sqrt{1+\frac{\lambda^2}{\rho_k}}} = \frac{1}{\sqrt{\prod_{k=0}^{\infty}\left(1+\frac{\lambda^2 t^2}{(k+\frac{1}{2})^2+\pi^2}\right)}}$$

$$= \frac{1}{\sqrt{\cosh(\lambda t)}}$$

The Schrödinger Equation

- Let us review the Schrödinger equation from quantum mechanics
  1. The "standard," time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\Psi(\mathbf{x}, t) = \left[\frac{-\hbar^2}{2m}\Delta + V(\mathbf{x}, t)\right]\Psi(\mathbf{x}, t) = \hat{H}(\mathbf{x}, t)\Psi$$

  2. We can make the equation dimensionless as

$$-i\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \left[\frac{1}{2}\Delta - V(\mathbf{r}, t)\right]\psi(\mathbf{x}, t) = H(\mathbf{x}, t)\psi$$

  3. We also are interested in the spectral properties of the time-independent problem

$$\left[\frac{1}{2}\Delta - V(\mathbf{x}, t)\right]\psi(\mathbf{x}, t) = H(\mathbf{x}, t)\psi = \lambda\psi$$

The Schrödinger and Bloch Equations

- We now arrive at the Bloch equation
  1. Consider transformation (analytic continuation) of the Schrödinger to imaginary time, $\tau = it$, this gives us the Bloch equation, but is sometimes also called the Schrödinger equation (going back to $u(\mathbf{x}, t)$)
  $$\frac{\partial u(\mathbf{x}, t)}{\partial \tau} = \frac{1}{2}\Delta u(\mathbf{x}, t) - V(\mathbf{x}, t)u(\mathbf{x}, t)$$
  2. The time dependent Bloch equation can be solved via separation of variables as
  $$u(\mathbf{x}, t) = U(\mathbf{x})T(t), \text{ and so we apply this to the Bloch equation}$$
  $$\frac{\partial u(\mathbf{x}, t)}{\partial t} = U(\mathbf{x})T'(t) = \left[\frac{1}{2}\Delta U(\mathbf{x}) - V(\mathbf{x}, t)U(\mathbf{x})\right] T(t)$$

The Schrödinger and Bloch Equations

3. Placing the time and space dependent on different sides of the equation gives

$$\frac{T'(t)}{T(t)} = \lambda = \frac{\left[\frac{1}{2}\Delta - V(\mathbf{x},t)\right] U(\mathbf{x})}{U(\mathbf{x})}, \text{where } \lambda \text{ is constant}$$

4. Thus we have that $T(t)$ and $U(\mathbf{x})$ satisfy the following equations

$$T'(t) - \lambda T(t) = 0,$$

$$\left[\frac{1}{2}\Delta - V(\mathbf{x},t)\right] U(\mathbf{x}) = \lambda U(\mathbf{x})$$

5. Thus the λ_j's and $\psi_j(\mathbf{x},t)$'s are eigenvalues and eigenfunctions of the above eigenvalue problem, and the solution by separation variables is

$$u(\mathbf{x},t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t}\psi_j(\mathbf{x}), \text{ where, } c_j = \int_{-\infty}^{\infty} u_0(\mathbf{x})\psi_j(\mathbf{x})\, d\mathbf{x}$$

The Schrödinger and Bloch Equations

- Let $\lambda = 1$, as $t \to \infty$, $E\left[e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds}\right] = \frac{1}{\sqrt{\cosh(t)}} \sim \sqrt{2}e^{-\frac{t}{2}}$ and
$$\lim_{t\to\infty} \frac{1}{t} \ln E\left[e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds}\right] = -\frac{1}{2}.$$
- Theorem: If $V(y) \to \infty$ as $|y| \to \infty$, then
$$\lim_{t\to\infty} \frac{1}{t} \ln E\left[e^{-\int_0^t V(\beta(s))\,ds}\right] = -\lambda_1,$$
where λ_1 is the lowest eigenvalue of the Bloch equation
$$\frac{1}{2}\psi''(y) - V(y)\psi(y) = \lambda\psi(y)$$

The Schrödinger and Bloch Equations

- Feynmann-Kac: Let V be measurable and bounded below, then the solution of the Bloch equation

$$u_t = \frac{1}{2}u_{xx} - V(x)u, \quad u(x,0) = u_0(x)$$

is $u(x,t) = E_x\left[e^{-\int_0^t V(\beta(s))\,ds}u_0(\beta(t))\right]$

- This equation is the imaginary time analog of the Schrödinger

$$\frac{1}{2}\psi''(y) - V(y)\psi(y) = \lambda\psi(y)$$

Equation

1. Special case: $V \equiv 0$:

$$E_x\left[u_0(\beta(t))\right] = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{2t}}\,dy = u(x,t)$$

Another special case

2. For $V(x) = \frac{x^2}{2}$, $u_0 \equiv 1$:

$$u(x,t) == E_x\left[e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds} \cdot 1\right] = E_0\left[e^{-\frac{1}{2}\int_0^t (\beta(s)+x)^2\,ds}\right]$$

$$= e^{-\frac{x^2 t}{2}} E\left[e^{-x\int_0^t \beta(s)\,ds - \frac{1}{2}\int_0^t \beta^2(s)\,ds}\right]$$

$$= e^{-\frac{x^2 t}{2}} E\left[e^{-x\sum_{k=0}^{\infty}\frac{\alpha_k}{\sqrt{\rho_k}}\int_0^t u_k(s)\,ds - \frac{1}{2}\sum_{k=0}^{\infty}\frac{\alpha_k^2}{\rho_k}}\right]$$

$$= e^{-\frac{x^2 t}{2}} \prod_{k=0}^{\infty} E\left[e^{-x\frac{\alpha_k}{\sqrt{\rho_k}}\int_0^t u_k(s)\,ds - \frac{1}{2}\frac{\alpha_k^2}{\rho_k}}\right]$$

$$= e^{-\frac{x^2 t}{2}} \prod_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x\frac{\alpha}{\sqrt{\rho_k}}\int_0^t u_k(s)\,ds - \frac{\alpha^2}{2}(1+\frac{1}{\rho_k})}\, d\alpha$$

$$= e^{-\frac{x^2 t}{2}} \frac{1}{\sqrt{\cosh(t)}} e^{\frac{x^2}{2}\int_0^t\int_0^t \sum_{k=0}^{\infty}\frac{u_k(s)u_k(\tau)}{\rho_k+1}\,ds\,d\tau}$$

- Define $R(s,\tau;-\lambda^2)$ such that

$$\min(s,\tau) = \lambda^2 \int_0^t \min(s,\xi) R(\xi,\tau;-\lambda^2)\, d\xi$$

Note that $R(s,\tau;-1) = -\sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k+1}$.

- Consider

$$-\sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k+\lambda^2} + \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k}$$

$$= \lambda^2 \int_0^t \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\xi)}{\rho_k} \sum_{l=0}^{\infty} \frac{u_l(\xi)u_l(\tau)}{\rho_k+\lambda^2}\, d\xi$$

- For $0 \leq s \leq t$ we have

$$R(s,\tau;-\lambda^2) = \begin{cases} -\frac{\cosh(\lambda(t-\tau))\sinh(\lambda s)}{\lambda\cosh(\lambda t)} & s \leq \tau \\ -\frac{\cosh(\lambda(t-s))\sinh(\lambda \tau)}{\lambda\cosh(\lambda t)} & s \geq \tau \end{cases}$$

- Thus

$$u(x,t) = \frac{1}{\cosh t} e^{-\frac{x^2}{2}\left(t+\int_0^t\int_0^t R(s,\tau;-1)\,ds\,d\tau\right)} = \frac{1}{\cosh t} e^{-\frac{x^2 \tanh t}{2}}$$

- Exercise: compute $u(x,t)$ for $V(x) = \frac{x^2}{2}$, $u_0(x) = x$. Hint: the solution is $u(x,t) = E_x\left[e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds}\beta(t)\right]$. Calculate

$$\tilde{u}(x,t,\lambda) = E_x\left[e^{\lambda\beta(t)-\frac{1}{2}\int_0^t \beta^2(s)\,ds}\right], \quad u(x,t) = \frac{d}{d\lambda}\tilde{u}(x,t,\lambda)\Big|_{\lambda=0}.$$

Proof of the Arcsin Law

- **Theorem:** Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with $E\left[X_i\right] = 0$, $Var(X_i) = 1$, and N_n is the number of partial sums $S_j = \sum_{i=1}^{j} X_i$ out of $S_1, \ldots, S_n$ which are ≥ 0:

$$\lim_{n\to\infty} P\left[\frac{N_n}{n} < \alpha\right] = \Sigma(\alpha) = \begin{cases} 0 & \alpha < 0 \\ \frac{2}{\pi} \arcsin\sqrt{\alpha} & 0 \leq \alpha \leq 1 \\ 1 & \alpha \geq 1 \end{cases}$$

- **Proof:** (Using the Feynman-Kac formula and Donsker's Invariance Principal) Define the random step function

$$X^{(n)}(\tau) = \begin{cases} \frac{S_1}{\sqrt{n}} & \tau = 0 \\ \frac{S_i}{\sqrt{n}} & \frac{i-1}{n} < \tau \leq \frac{i}{n} \end{cases}$$

The invariance principle states that for a large class of functionals $\mathcal{F}$ and $F \in \mathcal{F}$

$$\lim_{n\to\infty} P\left[F\left[X^{(n)}(\cdot)\right] \leq \alpha\right] = P_{BM}\left[F\left[\beta(\cdot)\right] \leq \alpha\right] \tag{2.2}$$

Proof of the Arcsin Law

- For example, let

$$F[\beta] = \int_0^t \frac{1+\mathrm{sgn}[\beta(s)]}{2}\, ds, \text{ where } \mathrm{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

- Then (2.2) says that

$$\lim_{n\to\infty} P\left[\frac{N_n}{n} \leq \alpha\right] = P_{BM}\left[\int_0^1 \frac{1+\mathrm{sgn}[\beta(s)]}{2}\, ds \leq \alpha\right]$$

  of the Brownian motion that is positive

- We drop the *BM* from the probabilities as it is understood

Proof of the Arcsin Law

- Let
$$\sigma(\alpha, t) = P\left[\int_0^t \frac{1+\mathrm{sgn}[\beta(s)]}{2}\, ds \le \alpha\right]$$
- Then for $\lambda > 0$ we can define the Laplace Transform/Moment Generating Function of $\sigma(\alpha, t)$
$$E\left[e^{-\lambda \int_0^t \frac{1+\mathrm{sgn}[\beta(s)]}{2}\, ds}\right] = \int_0^\infty e^{-\lambda\alpha}\, d\sigma(\alpha, t)$$
- Now define
$$u(x, t; \lambda) = E\left[e^{-\lambda \int_0^t \frac{1+\mathrm{sgn}[\beta(s)]}{2}\, ds}\delta(\beta(t) - x)\right]$$

Proof of the Arcsin Law

- By Feynman-Kac this is a solution to the following PDE

$$u(x,t;\lambda)_t = \frac{1}{2}u(x,t;\lambda)_{xx} - \lambda V(x)u(x,t;\lambda), \quad u(x,0;\lambda) = \delta(x)$$

$$\text{where } V(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- We also realize that

$$\int_{-\infty}^{\infty} u(x,t;\lambda)\,dx = \int_{-\infty}^{\infty} E\left[e^{-\lambda \int_0^t \frac{1+\text{sgn}[\beta(s)]}{2}\,ds}\delta(\beta(t)-x)\right]\,dx \overset{Fubini}{=}$$

$$E\left[\int_{-\infty}^{\infty} e^{-\lambda \int_0^t \frac{1+\text{sgn}[\beta(s)]}{2}\,ds}\delta(\beta(t)-x)\,dx\right] = E\left[e^{-\lambda \int_0^t \frac{1+\text{sgn}[\beta(s)]}{2}\,ds}\right] =$$

$$\int_0^{\infty} e^{-\lambda\alpha}\,d\sigma(\alpha,t)$$

Proof of the Arcsin Law

- It is know that $u(x,t;\lambda)$ also solves the following integral equation

$$u(x,t;\lambda) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} -$$

$$\lambda \int_0^t d\tau \int_{-\infty}^{\infty} d\xi V(\xi) u(\xi,\tau;\lambda) \frac{1}{\sqrt{2\pi(t-\tau)}} e^{\frac{-(x-\xi)^2}{2(t-\tau)}}$$

- Now we apply the heat equation operator, $\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$ to this

$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0 - \lambda V(x) u(x,t;\lambda)$$

- And we the Laplace transform of $u(x,t;\lambda)$

$$\Psi(x,s;\lambda) = \int_{-\infty}^{\infty} e^{-st} u(x,t;\lambda) dt$$

Proof of the Arcsin Law

- If we take the Laplace transform of the integral equation we get

$$\Psi(x,s;\lambda) = \frac{1}{\sqrt{2s}} e^{-\sqrt{2s}|x|}$$
$$-\lambda \int_{-\infty}^{\infty} d\xi\, V(\xi)\Psi(\xi,s;\lambda) \frac{1}{\sqrt{2s}} e^{-\sqrt{2s}|x-\xi|}$$

- This is equivalent to the following ordinary differential equation (ODE)

$$\frac{1}{2}\Psi''(x) - (s + \lambda V(x))\Psi(x) = 0, \Psi \to 0 \text{ as } |x| \to \infty$$

$$\Psi(x) \text{ and } \Psi'(x) \text{ is continuous at } x \neq 0, \text{ and } \Psi'(0^-) - \Psi'(0^+) = 2$$

Proof of the Arcsin Law

- The solution to the above ODE is

$$\Psi(x,s;\lambda) = \begin{cases} \frac{\sqrt{2}}{\sqrt{s+\lambda}+\sqrt{s}} e^{-\sqrt{2(s+\lambda)}x} & x \geq 0 \\ \frac{\sqrt{2}}{\sqrt{s+\lambda}+\sqrt{s}} e^{-\sqrt{2s}x} & x < 0 \end{cases}$$

- Thus we have that

$$\int_{-\infty}^{\infty} \Psi(x,s;\lambda)\,dx = \frac{1}{\sqrt{s(s+\lambda)}}$$

- So we have the following

$$\int_{-\infty}^{\infty} \Psi(x,s;\lambda)\,dx = \int_0^{\infty} e^{-st} \int_{-\infty}^{\infty} u(x,t;\lambda)\,dx\,ds =$$

$$\int_0^{\infty} e^{-st} \int_0^{\infty} e^{-\lambda\alpha}\,d\sigma(\alpha,t)\,ds = \frac{1}{\sqrt{s(s+\lambda)}}$$

Proof of the Arcsin Law

- The last line test us that we know the Laplace transform of

$$F(t) = \int_0^\infty e^{-\lambda\alpha}\, d\sigma(\alpha, t)\, ds$$

- The inverse Laplace transform of $\frac{1}{\sqrt{s(s+\lambda)}}$ tells us that

$$F(t) = e^{-\frac{\lambda t}{2}} I_o(\frac{\lambda t}{2}) = \int_0^\infty e^{-\lambda\alpha} \sigma'(\alpha, t)\, d\alpha$$

- Which is itself the Laplace transform of $\sigma'(\alpha, t)$, so we have

$$\sigma'(\alpha, t) = \begin{cases} \frac{1}{\pi\sqrt{\alpha(t-\alpha)}} & 0 < \alpha < t \\ 0 & \alpha > t \end{cases}$$

Proof of the Arcsin Law

- We now integrate the previous result

$$\int_{-\infty}^{\alpha} \sigma'(\bar{\alpha}, t)\, d\bar{\alpha} = \sigma(\alpha, t) = \begin{cases} 0 & 0 < \alpha \\ \frac{2}{\pi} \arcsin \sqrt{\frac{\alpha}{t}} & 0 < \alpha < t \\ 1 & \alpha > t \end{cases}$$

- Setting $t = 1$ we get the Arcsin Law

$$\sigma(\alpha, 1) = \Sigma(\alpha) = \begin{cases} 0 & 0 < \alpha \\ \frac{2}{\pi} \arcsin \sqrt{\frac{\alpha}{t}} & 0 < \alpha < 1 \\ 1 & \alpha > 1 \end{cases} \text{ Q. E. D.}$$

Another Wiener Integral

- We wish to compute the probability of

$$P\left\{\max_{0\le s\le t}\beta(s)\le\alpha\right\}$$

- By Donsker's Invariance Principal this is equal to

$$\lim_{n\to\infty}\left\{\max\left(\frac{S_1}{\sqrt{n}},\frac{S_2}{\sqrt{n}},\cdots,\frac{S_n}{\sqrt{n}}\right)\le\alpha\right\}=H(\alpha,t)$$

- Consider the step-function potential

$$V_\alpha(x)=\begin{cases}1 & x\ge\alpha\\ 0 & x<\alpha\end{cases}$$

- Since $\beta(\cdot)$ is a continuous function AE, if $\max_{0\le s\le t}\beta(s)\le\alpha$ then $V_\alpha(\beta(s))=0$ on a set of positive measure

Another Wiener Integral

- Consider the following Wiener integral

$$\lim_{\lambda \to \infty} E\left[e^{-\lambda \int_0^t V_\alpha(\beta(s))\,ds}\right] = H(\alpha, t)$$

- This is because the λ limit kills walks that exceed α and only count the walks that satisfy the condition
- for a fixed λ this is, by Feynman-Kac, the solution to

$$u(x,t;\lambda)_t = \frac{1}{2}u(x,t;\lambda)_{xx} - \lambda V(x)u(x,t;\lambda), \quad u(x,0;\lambda) = 1$$

$$\text{where } V(x) = \begin{cases} 1 & x \geq \alpha \\ 0 & x < \alpha \end{cases}$$

- The solution of the PDE is very similar to the solution of the PDE from the Arcsin Law, and is left to the reader

$$H(\alpha,t) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{\alpha}{\sqrt{t}}} e^{-\frac{u^2}{2}}\,du$$

Action Asympotics: A Heuristic for Wiener Integrals

- Von Neumann proved that there is no translationally invariant Haar measure in function space; Wiener measure is not translationally invariant
- Consider the following problem where we write our heuristic via a "flat" integral

$$E\left\{F[\beta]\right\} \text{``}=\text{''} \int F[\beta] e^{-\frac{1}{2}\int_0^t [\beta'(\tau)]^2 d\tau} \delta\beta$$

- Here we define the Action as

$$A[\beta] = -\frac{1}{2}\int_0^t \left[\beta'(\tau)\right]^2 d\tau$$

- This is obviously a heuristic, as BM is nondifferentiable AE

Action Asympotics: A Heuristic for Wiener Integrals

- Now consider computing the following with Action Asymptotics

$$E\left[e^{\frac{1}{\sqrt{\epsilon}}\int_0^t \beta(s)\,ds}\right]$$

- We first compute this using our standard techniques

$$\begin{aligned} E\left[e^{\frac{1}{\sqrt{\epsilon}}\int_0^t \beta(s)\,ds}\right] &= E\left[e^{\frac{1}{\sqrt{\epsilon}}\int_0^t \sum_{k=0}^{\infty}\frac{\alpha_k u_k(s)}{\sqrt{\rho_k}}\,ds}\right] = \\ E\left[e^{\frac{1}{\sqrt{\epsilon}}\sum_{k=0}^{\infty}\int_0^t \frac{\alpha_k}{\sqrt{\sqrt{\rho_k}}u_k(s)\,ds}}\right] &\overset{indep.}{=} \prod_{k=0}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{\alpha}{\sqrt{\epsilon\rho_k}}\int_0^t u_k(s)\,ds} e^{-\frac{\alpha^2}{2}}\,d\alpha \\ &= e^{\frac{t^3}{6\epsilon}} \end{aligned}$$

- And thus

$$\lim_{\epsilon\to 0}\epsilon \ln E\left[e^{\frac{1}{\sqrt{\epsilon}}\int_0^t \beta(s)\,ds}\right] = \frac{t^3}{6}$$

Action Asympotics: A Heuristic for Wiener Integrals

- Let's "derive" the action asymptotics heuristic with a construction due to Kac and Feynman by considering
$$Q(t) = E\left\{e^{-\int_0^t V(\beta(\tau))\,d\tau}\right\}$$
where $\beta(\cdot) \in C_0[0,t]$, and the expectation is taken w.r.t. Wiener measure
- Since we assume that $V(\cdot)$ is continuous and non-negative, and $\beta(\cdot) \in C_0[0,t]$) is continuous, $F(t)$ exists as $\int_0^t V(\beta(\tau))\,d\tau$ is measurable
- Now let us consider a discrete approximation of this Wiener integral by breaking it up into N sized time intervals of size t/N, which gives us $F(t)$ from bounded convergence and the Riemann summability
$$F(t) = \lim_{N\to\infty} E\left\{e^{-\frac{t}{N}\sum_{k=1}^{N} V(\beta(\frac{tk}{N}))}\right\}$$

Action Asympotics: A Heuristic for Wiener Integrals

- If we consider the expectation in the limit we can rewrite it as follows

$$\lim_{N\to\infty} E\left\{e^{-\frac{t}{N}\sum_{k=1}^{N} V(\beta(\frac{tk}{N}))}\right\} = \lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-h\sum_{k=1}^{N} V(\beta_k)} \times$$

$$P(0,\beta_1;h)P(\beta_1,\beta_2;h)\cdots P(\beta_{N-1},\beta_N;h)\,d\beta_1\,d\beta_2\cdots d\beta_N$$

where we have

1. $h = \frac{t}{N}$
2. $\beta_k = \beta(kh)$
3. $P(\beta_{k-1},\beta_k;h) = \frac{1}{\sqrt{2\pi h}} e^{-\frac{(\beta_k-\beta_{k-1})^2}{2h}}$

- This limit exists and is equal to the Wiener integral
- However, Feynman chose to rewrite the above as (suppressing the limit) with $\beta_0 = 0$

$$\frac{1}{(2\pi h)^{N/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-h\left\{\sum_{k=1}^{N} V(\beta(x_k))+\frac{1}{2}\sum_{k=1}^{N}\left(\frac{\beta_k-\beta_{k-1}}{h}\right)^2\right\}}\,d\beta_1\,d\beta_2\cdots d\beta_N$$

Action Asympotics: A Heuristic for Wiener Integrals

- If we look at the exponent in Feynman's we notice that

$$\left\{\sum_{k=1}^{N} V(\beta(x_k)) + \frac{1}{2}\sum_{k=1}^{N}\left(\frac{\beta_k - \beta_{k-1}}{h}\right)^2\right\} h \sim -\int_0^t \left\{\frac{1}{2}\left(\frac{d\beta}{d\tau}\right)^2 + V(\beta(\tau))\right\} d\tau$$

- This is the Hamiltonian the along the path, $\beta(\tau)$, and with the classical action along the path is

$$\int_0^t \left\{\frac{1}{2}\left(\frac{d\beta}{d\tau}\right)^2 - V(\beta(\tau))\right\} d\tau$$

- thus Feynman writes the above integral instead as

$$F(t) = E\left\{e^{-\int_0^t V(\beta(\tau))\,d\tau}\right\} = \int e^{-\left[\int_0^t \left\{\frac{1}{2}\left(\frac{d\beta}{d\tau}\right)^2 + V(\beta(\tau))\right\} d\tau\right]} d(\text{path})$$

Action Asympotics: A Heuristic for Wiener Integral

- How does $E\left[e^{\frac{1}{\epsilon}F[\sqrt{\epsilon}\beta]}\right]$ behave as $\epsilon \to 0$?
- We can approach this with Action Asymptotics

$$E\left[e^{\frac{1}{\epsilon}F[\sqrt{\epsilon}\beta]}\right] \text{ ``}=\text{''} \int e^{\frac{1}{\epsilon}F[\sqrt{\epsilon}\beta]} e^{-\frac{1}{2}\int_0^t [\beta'(s)]^2 ds} \delta\beta$$

- Now let $\sqrt{\epsilon}\beta = \omega$

$$\text{``}=\text{''} \int e^{\frac{1}{\epsilon}\left[F[\omega]-\frac{1}{2}\int_0^t [\omega'(s)]^2 ds\right]} \delta\beta$$

- Using Laplace asymptotics the above will behave like

$$e^{\frac{1}{\epsilon}\sup_{\omega\in C_0^*[0,t]}\left[F[\omega]-\frac{1}{2}\int_0^t [\omega'(s)]^2 ds\right]}$$

- Where the space $C_0^*[0,t]$ is made up functions, $\omega(t)$, with
  1. $\omega(t)$ continuous in $[0,t]$
  2. $\omega(0) = 0$
  3. $\omega'(t) \in L^2[0,t]$

Action Asympotics: Examples

- A conjecture using Action Asymptotics

$$\lim_{\epsilon \to 0} \epsilon \ln E\left[e^{\frac{1}{\epsilon} F[\sqrt{\epsilon}\beta]}\right] = \sup_{\omega \in C_0^*[0,t]} \left[F[\omega] - \frac{1}{2}\int_0^t [\omega'(s)]^2 ds\right]$$

- Consider $F[\beta] = \int_0^t \beta(s)\, ds$

$$E\left[e^{\frac{1}{\epsilon} F[\sqrt{\epsilon}\beta]}\right] = E\left[e^{\frac{1}{\sqrt{\epsilon}} \int_0^t \beta(s)\, ds}\right]$$

- From the conjecture we have that

$$\lim_{\epsilon \to 0} \epsilon \ln E\left[e^{\frac{1}{\sqrt{\epsilon}} \int_0^t \beta(s)\, ds}\right] = \sup_{\omega \in C_0^*[0,t]} \left[\int_0^t \omega(s)\, ds - \frac{1}{2}\int_0^t [\omega'(s)]^2 ds\right]$$

Action Asympotics: Examples

- From the calculus of variations we have that the Euler equation for following maximum principle is

$$\sup_{\omega \in C_0^*[0,t]} \left[\int_0^t \omega(s)\, ds - \frac{1}{2} \int_0^t [\omega'(s)]^2 ds \right] \Longrightarrow$$

1. $1 + \omega''(s) = 0$
2. $\omega(0) = 0$
3. $\omega'(t) = 0$

- The solution is $\omega(s) = -\frac{s^2}{2} + ts$ and $\omega'(s) = -s + t$ so

$$\int_0^t \left(-\frac{s^2}{2} + ts \right) ds - \frac{1}{2} \int_0^t [s - t]^2 ds = \frac{t^3}{6}$$

Brownian Scaling

- Recall some basic properties of the BM, $\beta(\cdot)$ and constant, c:
  1. $\beta(\tau) \sim N(0, \tau)$
  2. $\beta(c\tau) \sim N(0, c\tau)$
  3. $\sqrt{c}\beta(\tau) \sim N(0, c\tau)$
  4. $E[\beta(\tau)\beta(s)] = \min(\tau, s)$
  5. $E[\beta(c\tau)\beta(cs)] = c\min(\tau, s)$
  6. $E[\beta(c\tau)\beta(cs)] = E[\sqrt{c}\beta(\tau)\sqrt{c}\beta(s)] = cE[\beta(\tau)\beta(s)] = c\min(\tau, s)$
- Now consider the following

$$
\begin{aligned}
E\left[e^{\sup_{0\le s\le t}\beta(s)}\right] &= E\left[e^{\sup_{0\le \tau\le 1}\beta(t\tau)}\right] = \\
E\left[e^{\sup_{0\le \tau\le 1}\sqrt{t}\beta(\tau)}\right] &= E\left[e^{t\sup_{0\le \tau\le 1}\frac{1}{\sqrt{t}}\beta(\tau)}\right] = \\
E\left[e^{\frac{1}{\epsilon}\sup_{0\le \tau\le 1}\sqrt{\epsilon}\beta(\tau)}\right] & \qquad \text{using the substitution } t = \frac{1}{\epsilon}
\end{aligned}
$$

Action Asympotics: Examples

- So we now have that

$$\lim_{t\to\infty} \frac{1}{t} \ln E\left[e^{\sup_{0\le s\le t} \beta(s)}\right] = \lim_{\epsilon\to 0} \epsilon \ln E\left[e^{\frac{1}{\epsilon}\sup_{0\le\tau\le 1} \sqrt{\epsilon}\beta(\tau)}\right]$$

- By Action Asymptotics we have

$$\begin{aligned}\lim_{\epsilon\to 0} \epsilon \ln E\left[e^{\frac{1}{\epsilon}\sup_{0\le\tau\le 1} \sqrt{\epsilon}\beta(\tau)}\right] &= \sup_{\omega\in C_0^*[0,1]} \left[\sup_{0\le\tau\le 1} \omega(\tau) - \frac{1}{2}\int_0^1 [\omega'(\tau)]^2 d\tau\right] \\ &= \max_{a>0}\left[a - \frac{a^2}{2}\right] = \frac{1}{2}\end{aligned}$$

- The supremum comes on straight lines, that minimize arc-length i.e. the second term, so consider $\omega(\tau) = a\tau$, and $a = 1$ is the maximizer

Action Asympotics: Examples

- Consider a more complicated problem for Action Asymptotics is

$$\lim_{\epsilon\to 0} \frac{E\left[G\left(\sqrt{\epsilon}\beta(\cdot)\right) e^{\frac{1}{\epsilon}F\left(\sqrt{\epsilon}\beta(\cdot)\right)}\right]}{E\left[e^{\frac{1}{\epsilon}F\left(\sqrt{\epsilon}\beta(\cdot)\right)}\right]} \text{``} = \text{''}$$

$$\frac{\int E\left[G\left(\sqrt{\epsilon}\beta(\cdot)\right) e^{\frac{1}{\epsilon}F\left(\sqrt{\epsilon}\beta(\cdot)\right)-\frac{1}{2}\int_0^t\left[\beta'(s)\right]^2 ds}\right]\delta\beta}{\int E\left[e^{\frac{1}{\epsilon}F\left(\sqrt{\epsilon}\beta(\cdot)\right)-\frac{1}{2}\int_0^t[\beta'(s)]^2 ds}\right]\delta\beta} =$$

We now change variables with $x(\cdot) = \sqrt{\epsilon}\beta(\cdot)$

$$\frac{\int E\left[G\left(x(\cdot)\right) e^{\frac{1}{\epsilon}\left[F(x(\cdot))-\frac{1}{2}\int_0^t[x'(s)]^2 ds\right]}\right]\delta x}{\int E\left[e^{\frac{1}{\epsilon}\left[F(x(\cdot))-\frac{1}{2}\int_0^t[x'(s)]^2 ds\right]}\right]\delta x}$$

Action Asympotics: Examples

- As $\epsilon \to 0$ the exponential term goes to something like a "delta" function in function space and we get

$$= G\left[\omega^*(\cdot)\right] \text{ where } \omega^*(\cdot) = \underset{\omega \in C_0^*[0,t]}{\text{argsup}} \left[F[\omega] - A[\omega]\right]$$

- We now apply this to some PDE problems: Burger's Equation

$$\begin{aligned} u_t + u u_x &= \frac{\epsilon}{2} u_{xx}, \qquad -\infty \leq x \leq \infty, \quad t > 0 \\ u(x,0) &= u_0(x), \qquad \int_0^\infty u_0(\eta)\, d\eta = o(x^2) \text{ as } |x| \to \infty \end{aligned}$$

- We now apply the Hopf-Cole transformation, if we define the solution to Burger's equation $u(x,t) = -\epsilon \frac{v_x(x,t)}{v(x,t)} = -\epsilon \partial_x [\ln v(x,t)]$ then $v(x,t)$ satisfies

$$v_t = \frac{\epsilon}{2} v_{xx}, \quad v(x,0) = e^{-\frac{1}{\epsilon} \int_0^x u_0(\eta)\, d\eta}$$

Action Asympotics: Examples

- So by Feynman-Kac we can write the solution as

$$v(x,t;\epsilon) = \frac{1}{\sqrt{2\pi t\epsilon}} \int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\int_0^y u_0(\eta)\,d\eta} e^{-\frac{(x-y)^2}{2\epsilon t}}\, dy$$

- We now apply the Hopf-Cole transformation (taking the logarithmic derivative)

$$u(x,t;\epsilon) = \frac{\int_{-\infty}^{\infty} \frac{(x-y)}{t} e^{-\frac{1}{\epsilon}\left[\int_0^y u_0(\eta)\,d\eta + \frac{(y-x)^2}{2t}\right]}\, dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[\int_0^y u_0(\eta)\,d\eta + \frac{(y-x)^2}{2t}\right]}\, dy}$$

- Now let $F(y) = \int_0^y u_0(\eta)\,d\eta + \frac{(y-x)^2}{2t}$, this is the function that Action Asymptotics tells us to minimize (due to the negative sign)
- Note that $\lim_{|y|\to\infty} \frac{F(y)}{y^2} = \frac{1}{2t}$ by the assumptions, and so there is a minimum, $y(x,t) = \operatorname{argmin} F(y)$
- Hopf showed that if at (x,t) there is a single minimizer to $F(y)$ then

$$\lim_{\epsilon\to 0} u(x,t;\epsilon) = \frac{x - y(x,t)}{t} = u_0(y(x,t))$$

Action Asympotics: Examples

- Consider the related equation

$$\begin{aligned} u_t + uu_x &= \frac{\epsilon}{2} u_{xx} - V'(x), \qquad -\infty \leq x \leq \infty, \quad t > 0 \\ u(x,0) &= u_0(x), \qquad \int_0^\infty u_0(\eta)\, d\eta = o(x^2) \text{ as } |x| \to \infty \end{aligned}$$

- Again we use the Hopf-Cole transformation to get

$$v_t = \frac{\epsilon}{2} v_{xx} - \frac{1}{\epsilon} V'(x) v, \quad v(x,0) = e^{-\frac{1}{\epsilon}\int_0^x u_0(\eta)\, d\eta}$$

- And so we can write down the solution to the transformed equation via Feynman-Kac

$$v(x,t;\epsilon) = E_x \left[e^{-\frac{1}{\epsilon}\int_0^t V(\sqrt{\epsilon}\beta(s))\, ds - \frac{1}{\epsilon}\int_0^{\sqrt{\epsilon}\beta(t)} u_0(\eta)\, d\eta} \right]$$

$$= E_0 \left[e^{-\frac{1}{\epsilon}\left[\int_0^t V(\sqrt{\epsilon}\beta(s)+x)\, ds \int_0^{\sqrt{\epsilon}\beta(t)+x} u_0(\eta)\, d\eta\right]} \right]$$

Action Asympotics: Examples

- We now take apply the Hopf-Cole transformation and get

$$u(x,t;\epsilon) = \frac{E\left[G[\sqrt{\epsilon}\beta(\cdot)]e^{-\frac{1}{\epsilon}F[\sqrt{\epsilon}\beta(\cdot)]}\right]}{E\left[e^{-\frac{1}{\epsilon}F[\sqrt{\epsilon}\beta(\cdot)]}\right]} \text{ where we define}$$

$$F[\beta(\cdot)] = \int_0^t V(\sqrt{\epsilon}\beta(s))\,ds - \int_0^{\sqrt{\epsilon}\beta(t)} u_0(\eta)\,d\eta$$

$$G[\beta(\cdot)] = \int_0^t V'(\sqrt{\epsilon}\beta(s)+x)\,ds + u_0(\sqrt{\epsilon}\beta(t)+x)$$

Action Asympotics: Examples

- By Action Asymptotics we have that

$$\lim_{\epsilon\to 0} u(x,t;\epsilon) = G[\omega^*(\cdot)] \text{ where } \omega^*(\cdot) = \operatorname*{arginf}_{\omega \in C_0^*[0,t]} [F[\omega] + A[\omega]]$$

- If for (x,t) $\exists!$ minimizer, ω^*, then the limit exists and is

$$G[\omega^*(t)] = u(x,t) = \int_0^t V'(\omega^*(s)+x)\,ds + u_0(\omega^*(t)+x)$$

- Now consider the related variational problem

$$\inf_{\omega \in C_0^*[0,t]} \left[\int_0^t V(\omega(s)+x)\,ds \int_0^{\omega(t)+x} u_0(\eta)\,d\eta + \frac{1}{2}\int_0^t [\omega'(s)]^2\,ds \right]$$

- We refer to the functional to be minimized as $H[\omega(\cdot)]$

Action Asympotics: Examples

- To arrive derive an equivalent system via the Calculus of Variations we need to form the Frechet derivative, in the direction of the arbitrary function, Ψ, as follows

$$\delta H|_{\Psi} = \left.\frac{dH[\omega + h\Psi]}{dh}\right|_{h=0} = \int_0^t V'(\omega(s) + x)\Psi(s)\, ds + u_0(\omega(t) + x)\Psi(t)$$
$$+ \omega'(t)\Psi(t) - \int_0^t \omega''(s)\Psi(s)\, ds$$

- Note that the last two terms come from the following computation

$$J[\omega(\cdot)] \stackrel{\text{def}}{=} \frac{1}{2}\int_0^t [\omega'(s)]^2\, ds \implies \left.\frac{dJ[\omega + h\Psi]}{dh}\right|_{h=0}$$
$$= \frac{1}{2}\int_0^t [\omega'(s) + h\Psi'(s)]^2\, ds = \int_0^t [\omega'(s) + h\Psi'(s)]^2\, ds$$
$$= \int_0^t \omega'(s)\Psi'(s)\, ds = \int_0^t \omega'(s)\, d\Psi'(s)$$

Action Asympotics: Examples

- We now integrate by parts using the natural boundary conditions
  1. $\omega(0) = 0$
  2. $\omega'(0) = 0$

$$\int_0^t \omega'(s)\, d\Psi'(s) = \omega'(t)\Psi'(s) - \int_0^t \omega''(s)\Psi\, ds$$

- So the solution to this problem is
  1. $V'(\omega(s) + x) = \omega''(s)$ for $0 \leq s \leq t$
  2. $\omega(0) = 0$
  3. $\omega'(t) = -u_0(\omega(s) + x)$
- We can now apply this Hpof's result with $V \equiv 0$
  1. $\omega''(s) = 0$ for $0 \leq s \leq t$
  2. $\omega(0) = 0$
  3. $\omega'(t) = -u_0(\omega(s) + x)$
- The solution is then very simply
  1. $\omega(s) = cs$ for some constant, c
  2. $\omega'(s) = c = -u_0(ct + x)$
  3. Let $c = \frac{y(x,t)-x}{t} = -u_0(y(x,t))$ or $u_0(t(x,t)) = \frac{x-y(x,t)}{t}$
- With a unique $y(x,t)$ we get a unique $\omega^*(s) = \left(\frac{x-y(x,t)}{t}\right) s$

Action Asympotics

- We now consider some tools with the “flat integral"
- The Cameron-Martin Translation Formula

$$E\left\{F[\beta+y]\right\}, \text{ with } y \in C_0[0,t]$$

- We now use the “flat integral"

$$E\left\{F[\beta+y]\right\} \text{``}=\text{''} \int F[\beta+y]e^{-\frac{1}{2}\int_0^t[\beta'(s)]^2\,ds}\delta\beta, \text{ and let } \omega=\beta+y$$

$$\text{``}=\text{''} \int F[\omega]e^{-\frac{1}{2}\int_0^t[\omega'(s)-y'(s)]^2\,ds}\delta\omega$$

$$\text{``}=\text{''}\, e^{-\frac{1}{2}\int_0^t[y'(s)]^2\,ds} \int F[\omega]e^{+\int_0^t[\omega'(s)y'(s)]\,ds-\frac{1}{2}\int_0^t[\omega'(s)]^2\,ds}\delta\omega$$

$$\text{``}=\text{''}\, e^{-\frac{1}{2}\int_0^t[y'(s)]^2\,ds} E\left\{F[\beta]e^{\int_0^t y'(s)\,d\beta(s)}\right\}$$

- And so our result is that

$$E\left\{F[\beta+y]\right\} = e^{-\frac{1}{2}\int_0^t[y'(s)]^2\,ds} E\left\{F[\beta]e^{\int_0^t y'(s)\,d\beta(s)}\right\}, \text{ with } y \in C_0[0,t]$$

Local Time

- Spectral Theory:
- If $V(x) \geq 0$ and $V(x) \to 0$ as $|x| \to \infty$ then the eigenvalue problem

$$\frac{1}{2}\Psi''(x) - V(x)\Psi(x) = -\lambda\Psi(x)$$

1. Has discrete spectrum: $\lambda_1, \lambda_2, \cdots$
2. With corresponding eigenfunctions: $\Psi_1, \Psi_2, \cdots$

- Theorem (1949):

$$\lim_{t\to\infty} \frac{1}{t} E\left[e^{-\frac{1}{2}\int_0^t V(\beta(s))\,ds}\right] = -\lambda_1$$

Note: The expectation can start at any x due to ergodicity

- Proof We will first prove this using Feynman-Kac

$$u(x,t) = E_x\left[e^{-\frac{1}{2}\int_0^t V(\beta(s))\,ds}\right]$$

Local Time

- Satisfies the following PDE

$$u_t = \frac{1}{2}u_{xx} - V(x)u, \quad u(x,0) = 1$$

- By separation of variables we have

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t}\psi_j(x), \text{ where, } c_j = \int_{-\infty}^{\infty} u(x,0)\psi_j(y)\,dy$$

- But since $u(x,0) = 1$ we have that $c_j = \int_{-\infty}^{\infty} \psi_j(y)\,dy$, $\forall j \geq 0$, and so the two representations must be equal

$$u(x,t) = E_x\left[e^{-\frac{1}{2}\int_0^t V(\beta(s))\,ds}\right] = \sum_{j=1}^{\infty} e^{-\lambda_j t}\psi_j(x)\int_{-\infty}^{\infty}\psi_j(y)\,dy$$

- And so the largest eigenvalue, λ_1, controls the behavior

$$\lim_{t\to\infty}\frac{1}{t}E\left[e^{-\frac{1}{2}\int_0^t V(\beta(s))\,ds}\right] = -\lambda_1 \quad \square$$

Local Time

- We also have a variational representation of λ_1

$$\lambda_1 = \inf_{\substack{\Psi \in L^2 \\ ||\Psi||=1}} \left[\int_{-\infty}^{\infty} V(y)\Psi^2(y)\, dy + \frac{1}{2} \int_{-\infty}^{\infty} [\Psi'(y)]^2\, dy \right]$$

- Which has a corresponding Euler equation

$$\frac{1}{2}\Psi''(x) - V(x)\Psi(x) = -\lambda\Psi(x)$$

- We notice that in the Wiener integral representation, $E\left[e^{-\frac{1}{2}\int_0^t V(\beta(s))\, ds}\right]$, since the internal integral is in an negative exponential, the main contribution comes for paths that remain close to where $V(\cdot)$ is smallest, which leads us to dissect this problem as follows
- Let $\beta(s)$, $0 \leq s < \infty$; $\beta(0) = x$ be BM for $t > 0$ and consider the proportion of time that $\beta(\cdot)$ spends in a set $A \subset \mathbb{R}$

$$\ell_t(\beta(\cdot), \cdot) = \frac{1}{t} \int_0^t \chi_A(\beta(s))\, ds$$

Local Time

- Some properties of $L_t(\beta(\cdot), \cdot)$ with $t > 0$, x fixed, and $\beta(\cdot)$ a particular, fixed, path
  1. $L_t(\beta(\cdot), \cdot)$ is a countable additive, non-negative function
  2. $L_t(\beta(\cdot), \mathbb{R}) = 1$
  3. $L_t(\beta(\cdot), \cdot) : C_x[0, t] \to \mathcal{M}$, the space of probability measures on $\mathbb{R}$
- As a set function, $L_t(\beta(\cdot), \cdot)$ for fixed $x \in \mathbb{R}$ and $t > 0$ and for almost all $\beta(\cdot)$ has a density function which we call the normalized local time

$$\ell_t(\beta(\cdot), y) = \frac{1}{t} \int_0^t \delta(\beta(s) - y)\, dy \text{ and}$$

$$L_t(\beta(\cdot), A) = \int_{-\infty}^{\infty} \chi_A(y) \ell_t(\beta(\cdot), y)\, dy$$

- $\ell_t(\beta(\cdot), \cdot) \to 0$ as *Table* $\to \infty$ for compact A and almost every $\beta(\cdot)$
- Now consider the following representation

$$E_x\left[e^{-\int_0^t V(\beta(s))\, ds}\right] = E_x\left[e^{-t\int_{-\infty}^{\infty} V(y)\ell_t(\beta(\cdot), y)\, dy}\right]$$

Local Time

- For fixed $x \in \mathbb{R}$ and $t > 0$ we define a probability measure on $\mathcal{M}$, $Q_{x,t} = PL_t^{-1}$, as follows
- If $C \subset \mathcal{M}$ then we can write

$$Q_{x,t}(C) = P\left\{\beta(\cdot) \in C_x[0,\infty] : L_t(\beta(\cdot),\cdot) \in C\right\}$$

- $L_t(\beta(\cdot),\cdot)$ is an occupation measure so we can write

$$E_x\left[e^{-\int_0^t V(\beta(s))\,ds}\right] = E_x\left[e^{-t\int_{-\infty}^{\infty} V(y)\ell_t(\beta(\cdot),y)\,dy}\right] = E_x\left[e^{-t\int_{-\infty}^{\infty} V(y)\,dL_t(\beta(\cdot),y)}\right]$$

$$E_x^{Q_{x,t}}\left[e^{-t\int_{-\infty}^{\infty} V(y)\,\mu(dy)}\right] = E_x^{Q_{x,t}}\left[e^{-t\int_{-\infty}^{\infty} V(y)f(y)\,dy}\right]$$

- We define $\mathcal{F}$ as the space of probability density functions on $\mathbb{R}$, then this an expected value on $\mathcal{F}$
- To understand how the expected value on $\mathcal{F}$ behaves as $t \to \infty$, we need to understand how $Q_{x,t}$ and therefore also how $L_t(\beta(\cdot),A)$ behaves as $t \to \infty$

Local Time

- Long time behavior of local time measures
  1. $L_t(\beta(\cdot), A) \to 0$ as $t \to \infty$ for $A \subset \mathbb{R}$, compact, and AE $\beta(\cdot)$
  2. $\ell_t(\beta(\cdot), A) \to 0$ as $t \to \infty$ for $A \subset \mathbb{R}$, compact, and AE $\beta(\cdot)$ by the ergodic theorem for BM, if $\beta(\cdot)$ were not BM, then this would converge AE to the invariant measure
  3. $Q_{x,t}(C) \to 0$ as $t \to \infty$ if $C \subset \mathcal{M}$, $C \neq \mathcal{M}$, i.e. C is a reasonable set
- Theorem on Speed of Convergence: We first need to put the Levý topology on $\mathcal{F}$
  1. If $C \in \mathcal{F}$ is closed, then
$$\limsup_{t\to\infty} \frac{1}{t} \ln Q_{x,t}(C) \leq \inf_{f\in C} I(f)$$
  2. If $G \in \mathcal{F}$ is open, then
$$\liminf_{t\to\infty} \frac{1}{t} \ln Q_{x,t}(C) \geq \inf_{f\in G} I(f)$$
  3. Where
$$I(f) = \frac{1}{8}\int_{-\infty}^{\infty} \left\{ [f'(y)]^2/f(y) \right\} dy$$

Donsker-Varadhan Asympotics

- This is a simple case of what is referred to as “Donsker-Varadhan Asymptotics" and are a large deviation result
- An example, suppose $f(y) \sim N(0, \sigma^2)$, i.e. $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}$, then $f'(y) = -\frac{y}{\sigma^3\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}$ and $f'(y)^2 = \frac{y^2}{\sigma^6 2\pi} e^{-2\left(\frac{y^2}{2\sigma^2}\right)}$ and finally we have

$$I(f) = \frac{1}{8}\int_{-\infty}^{\infty} \left\{[f'(y)]^2/f(y)\right\} dy = \frac{1}{8}\frac{1}{\sigma^4}\int_{-\infty}^{\infty} \frac{y^2}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} dy = \frac{\sigma^2}{8\sigma^4} = \frac{1}{8\sigma^2}$$

Note: the last integral is the variance, σ^2, of a $N(0, \sigma^2)$ random variable

- We refer to the functional $I : \mathcal{F} \to [0, \infty]$ as the entropy, and roughly speaking

$$Q_{x,t}(f) \sim e^{-t \inf_{f \in A} I(f)} \text{ for “nice" } A$$

Donsker-Varadhan Asympotics

- Now let us apply the “Entropy Asymptotics" with the “Flat Integral"

$$E_x\left[e^{-\frac{1}{2}\int_0^t V(\beta(s))\,ds}\right] = E_x^{Q_{x,t}}\left[e^{-t\int_{-\infty}^{\infty} V(y)f(y)\,dy}\right] \text{ for } t \text{ large}$$

$$\text{“}=\text{”} \int e^{-t\int_{-\infty}^{\infty} V(y)f(y)\,dy}\, e^{-tI(f)}\,\delta f$$

$$\text{“}=\text{”} \int e^{-t\left[\int_{-\infty}^{\infty} V(y)f(y)\,dy + I(f)\right]}\,\delta f$$

- As $t \to \infty$ we use Laplace asymptotics to get

$$\lim_{t\to\infty}\frac{1}{t}\ln E_x\left[e^{-\frac{1}{2}\int_0^t V(\beta(s))\,ds}\right] = -\inf_{f\in y}\left[\int_{-\infty}^{\infty} V(y)f(y)\,dy + \frac{1}{8}\int_{-\infty}^{\infty}\frac{[f'(y)]^2}{f(y)}\,dy\right]$$

- Let $\sqrt{f(y)} = \Psi(y)$, then $\int_{-\infty}^{\infty}\Psi^2(y)\,dy = \int_{-\infty}^{\infty} f(y)\,dy = 1$ since $f(y)$ is a p.d.f., and so $\Psi(\cdot) \in L^2[-\infty,\infty]$ and $||\Psi|| = 1$

Donsker-Varadhan Asympotics

- We now transform the “Entropy Asymptotics" expression with some substitutions
  1. Let $\sqrt{f(y)} = \Psi(y)$, then $\int_{-\infty}^{\infty} \Psi^2(y)\, dy = \int_{-\infty}^{\infty} f(y)\, dy = 1$ since $f(y)$ is a p.d.f., and so $\Psi(\cdot) \in L^2[-\infty, \infty]$ and $||\Psi|| = 1$
  2. Also $\Psi'(y) = \frac{1}{2\sqrt{f(y)}} f'(y)$, and so $[\Psi'(y)]^2 = \frac{1}{4}\left(\frac{f'(y)^2}{f(y)}\right)$
- These allow us to write

$$-\inf_{f\in y}\left[\int_{-\infty}^{\infty} V(y)f(y)\, dy + \frac{1}{8}\int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)}\, dy\right] =$$

$$-\inf_{\substack{\Psi\in L^2\\ ||\Psi||=1}}\left[\int_{-\infty}^{\infty} V(y)\Psi^2(y)\, dy + \frac{1}{2}\int_{-\infty}^{\infty} [\Psi'(y)]^2\, dy\right] = -\lambda_1$$

- Theorem:Let $\Phi : \mathcal{F} \to \mathbb{R}$ be bounded and continuous then, by the “general structure theorem"

$$\lim_{t\to\infty} \frac{1}{t} \ln E_x^{Q_{x,t}}\left[e^{-t\Phi(f)}\right] = \lim_{t\to\infty} \frac{1}{t} \ln E_x\left[e^{-t\Phi(\ell_t(\beta(\cdot),\cdot))}\right] = -\inf_{f\in\mathcal{F}}[\Phi(f) + I(f)]$$

Donsker-Varadhan Asympotics

- This is more subtle than action asymptotics, for example consider

$$\lim_{t\to\infty} \frac{1}{t} \ln E_x^{Q_{x,t}} \left[e^{+t\Phi(f)} \right] = \sup_{f\in\mathcal{F}} [\Phi(f) - I(f)]$$

1. There is always a fight between the two terms in the supremum
2. In statistical mechanics we often consider $\alpha\Phi(f)$ and want to compute $\sup_{f\in\mathcal{F}} [\alpha\Phi(f) - I(f)] = g(\alpha)$, where α is a convex function of α
3. Them may be a critical value of α, call it α_0, where there is a phase transition, this is due to nonuniqueness in the f that maximized the functional

An Example Using Action and Entropy Asymptotics

- Now we will use “Entropy Asymptotics" to revisit a topic we have already considered
- Recall that

$$P\left\{\sup_{0\leq s\leq t}\beta(s)\leq\alpha\right\}=\sqrt{\frac{2}{\pi t}}\int_0^{\alpha}e^{-\frac{u^2}{2t}}\,du,\ \text{so that we also have}$$

$$E\left[e^{\sup_{0\leq s\leq t}\beta(s)}\right]=h(t)=\int_0^{\infty}e^{\alpha}\,dP\{\sup_{0\leq s\leq t}\beta(s)\leq\alpha\}=\int_0^{\infty}e^{\alpha}\sqrt{\frac{2}{\pi t}}e^{-\frac{\alpha^2}{2t}}\,d\alpha$$

$$\int_0^{\infty}e^{\alpha}\sqrt{\frac{2}{\pi t}}e^{-\frac{\alpha^2}{2t}}\,d\alpha=\sqrt{\frac{2}{\pi t}}\int_0^{\infty}e^{-\frac{(\alpha-t)^2}{2t}}e^{+\frac{t}{2}}\,d\alpha=\sqrt{\frac{2}{\pi}}e^{\frac{t}{2}}\int_{\sqrt{t}}^{\infty}e^{-\frac{u^2}{2}}\,du$$

with the substitution $u=\frac{\alpha-t}{\sqrt{t}}$

- Then we have

$$\lim_{t\to\infty}\frac{1}{t}\ln h(t)=\sqrt{\frac{2}{\pi}}e^{\frac{t}{2}}\int_{\sqrt{t}}^{\infty}e^{-\frac{u^2}{2}}\,du=\frac{1}{2}$$

An Example Using Action and Entropy Asymptotics

- First we turn the $t \to \infty$ limit into an $\epsilon \to 0$ limit

$$\lim_{t\to\infty} \frac{1}{t} \ln E\left[e^{\sup_{0\le s\le t} \beta(s)}\right] = \lim_{\epsilon\to 0} \epsilon \ln E\left[e^{\frac{1}{\epsilon} \sup_{0\le\tau\le 1} \sqrt{\epsilon}\beta(\tau)}\right]$$

- Recall that by Action Asymptotics we have

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon \ln E\left[e^{\frac{1}{\epsilon} \sup_{0\le\tau\le 1} \sqrt{\epsilon}\beta(\tau)}\right] &= \sup_{\omega\in C_0^*[0,1]} \left[\sup_{0\le\tau\le 1} \omega(\tau) - \frac{1}{2}\int_0^1 [\omega'(\tau)]^2 d\tau\right] \\ &= \max_{a>0} \left[a - \frac{a^2}{2}\right] = \frac{1}{2} \end{aligned}$$

- The supremum comes on straight lines, that minimize arc-length i.e. the second term, so consider $\omega(\tau) = a\tau$, and $a = 1$ is the maximizer

An Example Using Action and Entropy Asymptotics

- Now we solve the same problem using Entropy Asymptotics by using a result of Paul Levý that the following have the same probability distributions

$$P\left\{\sup_{0\le s\le t} \beta(s) \le \alpha\right\} = P\left\{t\ell_t(\beta(\cdot),0)\right\}$$

- Thus we have that

$$h(t) = E\left[e^{\sup_{0\le s\le t}\beta(s)}\right] = E\left[e^{t\ell_t(\beta(\cdot),0)}\right] = E\left[e^{t\Phi[\ell_t(\beta(\cdot),0)]}\right], \text{ where } \Phi[f] = f(0)$$

- So from Entropy Asymptotics we get

$$\lim_{t\to\infty}\frac{1}{t}\ln h(t) = \lim_{t\to\infty}\frac{1}{t}E\left[e^{t\Phi[\ell_t(\beta(\cdot),0)]}\right] = \sup_{f\in\mathcal{F}}\left[f(0) - \frac{1}{8}\int_{-\infty}^{\infty}\frac{[f'(y)]^2}{f(y)}\,dy\right]$$

- Recall that $f \in \mathcal{F}$ is a probability distribution, and so the maximizing family of functions (proven below) is $f_a(y) = ae^{-2a|y|}$

An Example Using Action and Entropy Asymptotics

- Recall that $f \in \mathcal{F}$ is a probability distribution, and so the maximizing family of functions (proven below) is $f_a(y) = ae^{-2a|y|}$
- We can write

$$f_a(y) = ae^{-2a|y|} = \begin{cases} ae^{-2ay} & y \geq 0 \\ ae^{2ay} & y < 0 \end{cases}, \text{ so } f_a'(y) = \begin{cases} -2a^2e^{-2ay} & y \geq 0 \\ 2a^2e^{2ay} & y < 0 \end{cases}, \text{ and so}$$

$$[f_a'(y)]^2 = \begin{cases} 4a^4e^{-4ay} & y \geq 0 \\ 4a^4e^{4ay} & y < 0 \end{cases} = 4a^4e^{-4a|y|}$$

- This gives us

$$\sup_{a>0}\left[f(0) - \frac{1}{8}\int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)}\,dy\right] = \sup_{a>0}\left[a - \frac{1}{8}\int_{-\infty}^{\infty} 4a^3e^{-2a|y|}\,dy\right]$$

$$= \sup_{a>0}\left[a - \frac{a^2}{2}\int_{-\infty}^{\infty} ae^{-2a|y|}\,dy\right] = \sup_{a>0}\left[a - \frac{a^2}{2}\right] = \frac{1}{2}, \text{ which occurs at } a = 1$$

An Example Using Action and Entropy Asymptotics

- Now we find the maximizing family of functions by the same transformation as before
  1. $\sqrt{f(y)} = \Psi(y)$ or $f(y) = \Psi^2(y)$, and so
  2. $f(0) = \Psi^2(0)$
  3. $\frac{1}{4}\left(\frac{f'(y)^2}{f(y)}\right) = [\Psi'(y)]^2$
- And so we obtain

$$\sup_{f\in\mathcal{F}}\left[f(0) - \frac{1}{8}\int_{-\infty}^{\infty}\frac{[f'(y)]^2}{f(y)}\,dy\right] = \sup_{\substack{\Psi\in L^2\\ ||\Psi||=1}}\left[\Psi^2(0) - \frac{1}{2}\int_{-\infty}^{\infty}[\Psi'(y)]^2\,dy\right]$$

- Let $\Psi(0) = a$ we get the following constrained Euler-Lagrange equation

$$\Psi''(y) - 2\lambda\Psi(y), \quad \Psi(0) = a$$

- This is maximized with a stretched version of $\Psi(y) = e^{-2|y|}$

Kac's Drum

- Let $\Omega \subset \mathbb{R}^2$ be an open domain with sufficiently smooth boundary, $\partial\Omega$, so that the following problem has a unique solution

$$\frac{1}{2}\Delta u + \lambda u = 0, \text{ with } u = 0 \text{ on } \partial\Omega$$

- Under these circumstances we know that
  1. $\exists \lambda_1 < \lambda_2 < \cdots$ a discrete spectrum
  2. $\exists u_1(x, y) < u_1(x, y) < \cdots$ corresponding normalized eigenfunctions
- Consider

$$C(\lambda) = \sum_{\lambda_j < \lambda} 1 = \text{ \# of eigenvalues } < \lambda$$

- $C(\lambda)$ is an increasing function in λ, and Hermen Weyl proved that

$$C(\lambda) \sim \frac{|\Omega|\lambda}{2\pi} \text{ as } \lambda \to \infty$$

- Additionally, Carlemann proved that

$$\sum_{\lambda_j < \lambda} u(x, y) \sim \frac{\lambda}{2\pi}, \forall\, (x, y) \in \Omega \text{ as } \lambda \to \infty$$

Kac's Drum

- Now consider starting a BM at $(x_0, y_0) \in \Omega$
- Let $p(x_0, y_0, x, y, t)$ be the probability density function of a 2D BM starting at (x_0, y_0) reaching (x, y) at time t without hitting $\partial\Omega$
- Einstein-Smoluchowski: Then $p(x_0, y_0, x, y, t)$ is the solution to

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta p \text{ in } \Omega$$
$$p = 0 \text{ on } \partial\Omega, \quad \forall t > 0$$

- We note that as $t \to 0$

$$\int_\Omega g(x, y)p(x_0, y_0, x, y, t)\, dx\, dy \to g(x_0, y_0)$$

- Assume we can find p using separation of variables: $p(x_0, y_0, x, y, t) = T(t)U(x, y)$, then

$$T'U = \frac{T}{2}\Delta U, \quad U = 0 \text{ on } \partial\Omega, \quad \forall t > 0$$
$$\frac{T'}{T} = \frac{\Delta U}{2} = -\lambda \text{ yields}$$
$$T(t) = e^{-\lambda t}, \text{ and } U = \text{ the eigenfunction corresponding to } \lambda$$

Kac's Drum

- So this means that we can write explicitly

$$p(x_0, y_0, x, y, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j(x_0, y_0) u_j(x, y), \text{ and so we know}$$

$$p(x_0, y_0, x_0, y_0, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0, y_0)$$

- Let $p^*(x_0, y_0, x, y, t)$ be the probability density function of unrestricted 2D BM starting at (x_0, y_0) reaching (x, y) at time t

$$p^*(x_0, y_0, x, y, t) = \frac{1}{2\pi t} e^{-\frac{(x-x_0)^2}{2t} - \frac{(y-y_0)^2}{2t}}$$

- Thus we conclude that

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0, y_0) \sim p^*(x_0, y_0, x, y, t) \sim \frac{1}{2\pi t} \text{ as } t \to 0$$

Kac's Drum

- Karamata Tauberian Theorem: Consider

$$f(t) = \int_0^\infty e^{-\lambda t}\, d\alpha(\lambda), \text{ and assume}$$

1. The above Laplace-Stiltje's transform exists
2. $\alpha(\lambda)$ is non-decreasing on $(0, \infty)$

- If $f(t) \sim At^{-\gamma}$ as $t \to 0$ for A and γ constants then

$$\alpha(\lambda) \sim \frac{A\lambda^\gamma}{\Gamma(\gamma+1)} \text{ as } \lambda \to \infty (\lambda \to 0)$$

- We now apply the Karamata Tauberian Theorem to

$$f(t) = \int_0^\infty e^{-\lambda t}\, d\alpha(\lambda) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0, y_0), \text{ where } \alpha(\lambda) = \sum_{\lambda_j < \lambda} u_j^2(x_0, y_0)$$

- We know $f(t) \sim \frac{1}{2\pi t}$ as $t \to 0$, and so $\alpha(\lambda) \sim \frac{\lambda}{2\pi}$ as $\lambda \to \infty$
- By integrating this over Ω we get Weyl's theorem

Probabilistic Potential Theory

1. Let $\Omega \in \mathbb{R}^3$ be a bounded closed domain
2. Let $\mathbf{r}(t) \in \mathbb{C}$ be a continuous function starting at the origin
3. Let $\chi_\Omega(\cdot)$ be the indicator function of Ω

- Consider the following functional on $\mathbb{C}$

$$T_\Omega(\mathbf{y}, \mathbf{r}(\cdot)) = \int_0^\infty \chi_\Omega(\mathbf{y} + \mathbf{r}(\tau))\, d\tau, \quad \mathbf{y} \in \mathbb{R}^3$$

- This functional is the total occupations time of $\mathbf{r}(\cdot)$, a 3D BM, in Ω translated by $\mathbf{y}$
- Now impose Wiener measure on $\mathbb{C}$ and consider the following Wiener integral

$$E\left\{T_\Omega(\mathbf{y}, \mathbf{r}(\cdot))\right\} = \int_0^\infty P\left\{\mathbf{y} + \mathbf{r}(\tau) \in \Omega\right\}\, d\tau$$

- Note that because we are using Wiener measure we know

$$P\left\{\mathbf{y} + \mathbf{r}(\tau) \in \Omega\right\} = \frac{1}{(2\pi\tau)^{3/2}} \int_0^\infty e^{-\frac{|\mathbf{r}-\mathbf{y}|^2}{2\tau}}\, d\mathbf{r}$$

Probabilistic Potential Theory

- We now use Fubini's theorem to exchange the order of integration

$$E\left\{T_\Omega(\mathbf{y},\mathbf{r}(\cdot))\right\} = \int_\Omega d\mathbf{r} \int_0^\infty \frac{1}{(2\pi\tau)^{3/2}} e^{-\frac{|\mathbf{r}-\mathbf{y}|^2}{2\tau}}\, d\tau$$

$$= \frac{1}{2\pi}\int_\Omega \frac{d\mathbf{r}}{|\mathbf{r}-\mathbf{y}|} < \infty \text{ in } \mathbb{R}^3$$

- We see that in $\mathbb{R}^3$ AE BM path starting at $\mathbf{y}$ spends a finite amount of time in Ω
- Now consider the kth moment of the occupation time

$$E\left\{T_\Omega^k(\mathbf{y},\mathbf{r}(\cdot))\right\} = \frac{k!}{(2\pi)^k}\int_\Omega \overset{[k]}{\cdots} \int_\Omega \frac{d\mathbf{r}_1}{|\mathbf{r}_1-\mathbf{y}|}\frac{d\mathbf{r}_2}{|\mathbf{r}_2-\mathbf{r}_1|}\cdots\frac{d\mathbf{r}_k}{|\mathbf{r}_k-\mathbf{r}_{k-1}|} \quad k=1,2,\cdots$$

- We focus on the second moment, $k=2$

$$E\left\{T_\Omega^2(\mathbf{y},\mathbf{r}(\cdot))\right\} = \int_0^\infty\int_0^\infty P\left\{\mathbf{y}+\mathbf{r}(\tau_1)\in\Omega\right\} P\left\{\mathbf{y}+\mathbf{r}(\tau_2)\in\Omega\right\}\, d\tau_1\, d\tau_2$$

Probabilistic Potential Theory

- We focus on the second moment, $k = 2$

$$E\left\{T_\Omega^2(\mathbf{y}, \mathbf{r}(\cdot))\right\} = \int_0^\infty \int_0^\infty P\left\{\mathbf{y} + \mathbf{r}(\tau_1) \in \Omega\right\} P\left\{\mathbf{y} + \mathbf{r}(\tau_2) \in \Omega\right\} \, d\tau_1 \, d\tau_2$$

$$= 2 \iint_{0 \le \tau_1 < \tau_2 < \infty} d\tau_1 d\tau_2 \int_\Omega \int_\Omega \frac{1}{(2\pi\tau_1)^{3/2}} e^{-\frac{|\mathbf{r}_1 - \mathbf{y}|^2}{2\tau}} \frac{1}{[2\pi(\tau_2 - \tau_1)]^{3/2}} e^{-\frac{|\mathbf{r}_2 - \mathbf{r}_1|^2}{2(\tau_2 - \tau_1)}} d\mathbf{r}_1 d\mathbf{r}_2$$

$$= \frac{2}{(2\pi)^2} \int_\Omega \int_\Omega \frac{d\mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{y}|} \frac{d\mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

- The formula for the kth moment suggests that we should consider the following eigenvalue problem

$$\frac{1}{2\pi} \int_\Omega \frac{\phi(\boldsymbol{\rho})}{|\mathbf{r} - \boldsymbol{\rho}|} \, d\boldsymbol{\rho} = \lambda \phi(\mathbf{r}), \quad \mathbf{r} \in \Omega$$

Probabilistic Potential Theory

- The integral kernel in the eigenvalue problem is Hilbert-Schmidt
  1. Since the single integral is convergent, we have
  $$\int_\Omega \int_\Omega \frac{1}{|\mathbf{r}-\boldsymbol{\rho}|^2}\, d\mathbf{r}\, d\boldsymbol{\rho} < \infty$$
  2. We also need to show that the kernel is positive definite:
  $$\int_\Omega \int_\Omega \frac{\phi(\mathbf{r})\phi(\boldsymbol{\rho})}{|\mathbf{r}-\boldsymbol{\rho}|}\, d\mathbf{r}\, d\boldsymbol{\rho} > 0 \quad \forall \phi(\boldsymbol{\rho}) \neq 0 \text{ in } L^2(\Omega)$$

  Note that:
  $$\frac{1}{2\pi}\frac{1}{|\mathbf{r}-\boldsymbol{\rho}|} = \int_0^\infty \frac{1}{(2\pi\tau)^{3/2}} e^{-\frac{|\mathbf{r}-\mathbf{y}|^2}{2\tau}}\, d\tau =$$
  $$\int_0^\infty d\tau \frac{1}{(2\pi\tau)^{3/2}} \frac{\tau^{3/2}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i\zeta\cdot(\mathbf{r}-\boldsymbol{\rho})} e^{\frac{-|\zeta|^2\tau}{2}}\, d\zeta =$$
  $$\frac{1}{(2\pi)^3}\int_0^\infty d\tau \int_{\mathbb{R}^3} d\zeta e^{i\zeta\cdot(\mathbf{r}-\boldsymbol{\rho})} e^{\frac{-|\zeta|^2\tau}{2}}$$

Probabilistic Potential Theory

- So

$$\int_\Omega \int_\Omega \frac{\phi(\mathbf{r})\phi(\boldsymbol{\rho})}{|\mathbf{r}-\boldsymbol{\rho}|}\, d\mathbf{r}\, d\boldsymbol{\rho} =$$

$$\frac{1}{(2\pi)^3} \int_0^\infty d\tau \int_{\mathbb{R}^3} d\zeta e^{\frac{-|\zeta|^2\tau}{2}} \left| \int_\Omega \phi(\boldsymbol{\rho}) e^{i\zeta\cdot\boldsymbol{\rho}}\, d\boldsymbol{\rho} \right|^2 > 0, \ \forall \phi(\boldsymbol{\rho}) \neq 0 \text{ in } L^2(\Omega)$$

- With the kernel being Hilbert-Schmidt, we know that the integral equation has
  1. Discrete spectrum: $\lambda_1, \lambda_2, \cdots$
  2. With corresponding eigenfunctions that form a complete, orthonormal basis for $L^2(\Omega)$
- Lemma:

$$\frac{1}{k!} E\left\{ T_\Omega^k (\mathbf{y}, \mathbf{r}(\cdot)) \right\} = \sum_{j=1}^\infty \lambda_j^{k-1} \int_\Omega \phi_j(\mathbf{r})\, d\mathbf{r} \frac{1}{2\pi} \int_\Omega \frac{\phi_j(\boldsymbol{\rho})}{|\boldsymbol{\rho}-\mathbf{y}|}\, d\boldsymbol{\rho}$$

  1. This holds for all $y \in \mathbb{R}^3$
  2. If $y \in \Omega$, then we note that

$$\frac{1}{2\pi} \int_\Omega \frac{\phi_j(\boldsymbol{\rho})}{|\boldsymbol{\rho}-\mathbf{y}|}\, d\boldsymbol{\rho} = \lambda_j \phi_j(\mathbf{y})$$

Probabilistic Potential Theory

- Proof: Recall that

$$\frac{1}{k!}E\left\{T_\Omega^k(\mathbf{y},\mathbf{r}(\cdot))\right\} = \frac{1}{(2\pi)^k}\int_\Omega \overset{[k]}{\cdots} \int_\Omega \frac{d\mathbf{r}_1}{|\mathbf{r}_1-\mathbf{y}|}\frac{d\mathbf{r}_2}{|\mathbf{r}_2-\mathbf{r}_1|}\cdots\frac{d\mathbf{r}_k}{|\mathbf{r}_k-\mathbf{r}_{k-1}|}$$

- We recognize this as an iterated integral equation of the form

$$a(\mathbf{y},\mathbf{r}_1)a(\mathbf{r}_1,\mathbf{r}_2)\cdots a(\mathbf{r}_{k-1},\mathbf{r}_k)$$

- We can then rewrite this using Mercer's theorem representation of the kernel of the integral operator

$$\frac{1}{|\boldsymbol{\rho}-\mathbf{y}|} = \sum_{j=1}^{\infty}\lambda_j\phi_j(\boldsymbol{\rho})\phi_j(\mathbf{y})$$

- Next we apply Mercer's theorem only to the terms not involving **y** to get

$$\frac{1}{k!}E\left\{T_\Omega^k(\mathbf{y},\mathbf{r}(\cdot))\right\} = \frac{1}{2\pi}\int_\Omega \frac{1}{|\mathbf{r}_1-\mathbf{y}|}\int_\Omega \sum_{j=1}^{\infty}\lambda_j^{k-1}\phi_j(\mathbf{r}_1)\phi_j(\mathbf{r}_k)\,d\mathbf{r}_1\,d\mathbf{r}_k$$

Probabilistic Potential Theory

- To review we have that

$$\frac{1}{k!}E\left\{T_\Omega^k(\mathbf{y},\mathbf{r}(\cdot))\right\} = \begin{cases} \sum_{j=1}^{\infty}\lambda_j^{k-1}\int_\Omega \phi_j(\mathbf{r})\,d\mathbf{r}\frac{1}{2\pi}\int_\Omega \frac{\phi_j(\boldsymbol{\rho})}{|\boldsymbol{\rho}-\mathbf{y}|}\,d\boldsymbol{\rho}, & \mathbf{y}\in\mathbb{R}^3 \\ \sum_{j=1}^{\infty}\lambda_j^{k}\int_\Omega \phi_j(\mathbf{r})\phi_j(\mathbf{y})\,d\mathbf{r}, & \mathbf{y}\in\Omega \end{cases}$$

- Now let us consider the moment generation function (Laplace transform) with $z\in\mathbb{C}$

$$E\left\{e^{zT_\Omega(\mathbf{y},\mathbf{r}(\cdot))}\right\} = \sum_{k=0}^{\infty}\frac{z^k}{k!}E\left\{T_\Omega^k(\mathbf{y},\mathbf{r}(\cdot))\right\}$$

- Now we use the above lemma to get

$$= 1 + \frac{z}{2\pi}\sum_{j=1}^{\infty}\left(\frac{1}{1-\lambda_j z}\right)\int_\Omega \phi_j(\mathbf{r})\,d\mathbf{r}\int_\Omega \frac{\phi_j(\boldsymbol{\rho})}{|\boldsymbol{\rho}-\mathbf{y}|}\,d\boldsymbol{\rho}$$

1. This series converges if $|z| < \frac{1}{\lambda_{max}}$
2. The moment generating function is analytic if $\Re\{z\} < 0$ since $T_\Omega \geq 0$
3. The last series is analytic for $\Re\{z\} < 0$, so by analytic continuation this identity holds with $\Re\{z\} <$

Probabilistic Potential Theory

- Let $u > 0$ and define

$$h(\mathbf{y}, u) = E\left\{e^{-uT_\Omega(\mathbf{y},\mathbf{r}(\cdot))}\right\} = 1 - \frac{u}{2\pi}\sum_{j=1}^{\infty}\left(\frac{1}{1+\lambda_j u}\right)\int_\Omega \phi_j(\mathbf{r})\,d\mathbf{r}\int_\Omega \frac{\phi_j(\boldsymbol{\rho})}{|\boldsymbol{\rho}-\mathbf{y}|}\,d\boldsymbol{\rho} \qquad (*)$$

- This series converges on compact sets in $\mathbb{C}$ because

1. $$\frac{1}{1+\lambda_j u} < 1$$

2. $$\left(\sum_{j=1}^{\infty}\int_\Omega \phi_j(\mathbf{r})\,d\mathbf{r}\int_\Omega \frac{\phi_j(\boldsymbol{\rho})}{|\boldsymbol{\rho}-\mathbf{y}|}\,d\boldsymbol{\rho}\right)^2 \leq \sum_{j=1}^{\infty}\left(\int_\Omega \phi_j(\mathbf{r})\,d\mathbf{r}\right)^2\sum_{j=1}^{\infty}\left(\int_\Omega \frac{\phi_j(\boldsymbol{\rho})}{|\boldsymbol{\rho}-\mathbf{y}|}\,d\boldsymbol{\rho}\right)^2 = |\Omega|\int_\Omega \frac{d\boldsymbol{\rho}}{|\boldsymbol{\rho}-\mathbf{y}|} < \infty$$

- This gives uniform convergence via the Weierstrass M-test and thus this is also analytic

Probabilistic Potential Theory

- If $\mathbf{y} \in \Omega$ then we get

$$h(\mathbf{y}, u) = 1 - \sum_{j=1}^{\infty} \left(\frac{\lambda_j u}{1 + \lambda_j u} \right) \int_{\Omega} \phi_j(\mathbf{r}) \, d\mathbf{r} \, \phi_j(\mathbf{y})$$

- And so we can multiply both sides by $\frac{1}{2\pi |\mathbf{y}-\mathbf{r}|}$ and integrate over Ω

$$\frac{1}{2\pi} \int_{\Omega} \frac{h(\mathbf{y}, u) \, d\mathbf{y}}{|\mathbf{y} - \mathbf{r}|} = \frac{1}{2\pi} \int_{\Omega} \frac{d\mathbf{y}}{|\mathbf{y} - \mathbf{r}|} - \sum_{j=1}^{\infty} \left(\frac{\lambda_j u}{1 + \lambda_j u} \right) \int_{\Omega} \phi_j(\boldsymbol{\rho}) \, d\boldsymbol{\rho} \frac{1}{2\pi} \int_{\Omega} \frac{\phi_j(\mathbf{y}) \, d\mathbf{y}}{|\mathbf{y} - \mathbf{r}|}$$

- But we know that

$$\frac{1}{2\pi} \int_{\Omega} \frac{d\mathbf{y}}{|\mathbf{y} - \mathbf{r}|} = \sum_{j=1}^{\infty} \int_{\Omega} \phi_j(\boldsymbol{\rho}) \, d\boldsymbol{\rho} \frac{1}{2\pi} \int_{\Omega} \frac{\phi_j(\mathbf{y}) \, d\mathbf{y}}{|\mathbf{y} - \mathbf{r}|}$$

- Thus we cane write that

$$\frac{1}{2\pi} \int_{\Omega} \frac{h(\mathbf{y}, u) \, d\mathbf{y}}{|\mathbf{y} - \mathbf{r}|} = \sum_{j=1}^{\infty} \left(\frac{1}{1 + \lambda_j u} \right) \int_{\Omega} \phi_j(\boldsymbol{\rho}) \, d\boldsymbol{\rho} \frac{1}{2\pi} \int_{\Omega} \frac{\phi_j(\mathbf{y}) \, d\mathbf{y}}{|\mathbf{y} - \mathbf{r}|}$$

Probabilistic Potential Theory

- We recognize the left hand side of the previous equation from (*), and so we use this ro rewrite this as
$$\frac{1}{2\pi}\int_\Omega \frac{h(\mathbf{y},u)\,d\mathbf{y}}{|\mathbf{y}-\mathbf{r}|} = \frac{1}{u}\left(1-h(\mathbf{r},u)\right), \quad \forall \mathbf{r}\in\mathbb{R}^3$$
- Moreover, if we rename variables we get
$$\frac{1}{2\pi}\int_\Omega \frac{h(\boldsymbol{\rho},u)\,d\boldsymbol{\rho}}{|\mathbf{y}-\boldsymbol{\rho}|} = \frac{1}{u}\left(1-h(\mathbf{y},u)\right), \quad \forall \mathbf{y}\in\mathbb{R}^3 \tag{**}$$
- We now make some important observations
  1. From (*) we see that if $\mathbf{y}\notin\Omega$ then $h(\mathbf{y},u)$ is harmonic in $\mathbf{y}$, and the series in (*) converges uniformly on compact Ω's
  2. Again from (*) we get
$$h(\mathbf{y},u) > 1-\frac{u}{2\pi}\left\{\sum_{j=1}^{\infty}\left(\int_\Omega \phi_j(\boldsymbol{\rho})\,d\boldsymbol{\rho}\right)^2\right\}^{1/2}\left\{\sum_{j=1}^{\infty}\left(\int_\Omega \frac{\phi_j(\boldsymbol{\rho})}{|\boldsymbol{\rho}-\mathbf{y}|}\,d\boldsymbol{\rho}\right)^2\right\}^{1/2}$$
$$> 1-\frac{u}{2\pi}|\Omega|^{1/2}\left(\int_\Omega \frac{d\boldsymbol{\rho}}{|\boldsymbol{\rho}-\mathbf{y}|}\right)^{1/2}$$

Probabilistic Potential Theory

3. So we now know that $0 \leq h(\mathbf{y}, u) \leq 1$, and so
$$\lim_{u \nearrow \infty} h(\mathbf{y}, u) = 1 \tag{***}$$
4. And for from Courant-Hilbert II, pp. 245–246
$$\Delta \left(\int_{\Omega} \frac{h(\mathbf{y}, u)\, d\mathbf{y}}{|\mathbf{y} - \mathbf{r}|} \right) = -4\pi h(\mathbf{y}, u)$$

- Now apply the Laplacian to both sides of (**) to get
$$-2h(\mathbf{y}, u) = -\frac{1}{u} \Delta h(\mathbf{y}, u)$$
or we get
$$\frac{1}{2} \Delta h(\mathbf{y}, u) - uh(\mathbf{y}, u) = 0, \quad \mathbf{y} \in \Omega$$
- Now consider $\mathcal{U}(\mathbf{y}) = \lim_{u \nearrow \infty} (1 - h(\mathbf{y}, u)) = P\left\{T_{\Omega}(\mathbf{y}, \mathbf{r}(\cdot)) > 0\right\}$, this is the capacitory potential (capacitance) and follows easily from the definition of the moment generating function

Probabilistic Potential Theory

- Example: Let Ω be a sphere of radius 1 centered at the origin
  1. $h(\mathbf{y}, u)$ is clearly spherically symmetric
  2. $h(\mathbf{y}, u)$ is harmonic outside Ω, so we have
  $$h(\mathbf{y}, u) = \frac{\alpha(u)}{|\mathbf{y}|} + \beta(u), \quad \mathbf{y} \notin \Omega$$
  3. From (***) we see that $\beta(u) = 1$ and so $h(\mathbf{y}, u) = \frac{\alpha(u)}{|\mathbf{y}|} + 1$ for $\mathbf{y} \in \Omega$
  4. We also know that for $\mathbf{y} \in \Omega$ we have
  $$h(\mathbf{y}, u) = \gamma(u) \frac{\sinh(\sqrt{2u}\,|\mathbf{y}|)}{|\mathbf{y}|}$$
  5. If we substitute this into the equation (**) we get that $\gamma(u) = \frac{1}{\sqrt{2u}\cosh(a\sqrt{2u})}$
  6. $h(\mathbf{y}, u)$ is continuous $\forall \mathbf{y}$ so from the uniform convergence of the series, and so
  $$\frac{\alpha(u)}{a} + 1 = \frac{1}{\sqrt{2u}} \frac{\sinh(\sqrt{2u}a)}{\cosh(\sqrt{2u}a)} \frac{1}{a}$$
  to finally give us
  $$h(\mathbf{y}, u) = \begin{cases} 1 - \frac{1}{|\mathbf{y}|}\left(1 - \frac{\tanh(a\sqrt{2u})}{a\sqrt{2u}}\right), & \mathbf{y} \notin \Omega \\ \frac{\sinh(\sqrt{2u}|\mathbf{y}|)}{\sqrt{2u}\cosh(\sqrt{2u}a)|\mathbf{y}|}, & \mathbf{y} \in \Omega \end{cases}$$

Probabilistic Potential Theory

- Recall that

$$\mathcal{U}(\mathbf{y}) = \lim_{u \nearrow \infty} (1 - h(\mathbf{y}, u)) = P\left\{ T_{S(0,a)}(\mathbf{y}, \mathbf{r}(\cdot)) > 0 \right\} = \begin{cases} \frac{a}{|\mathbf{y}|}, & \mathbf{y} \notin \Omega \\ 1, & \mathbf{y} \in \Omega \end{cases}$$

- This is the capacitory potential of $S(0, a)$
- Now back to the general case, $\forall \mathbf{y} \in \mathbb{R}^3$ we have

$$1 - E\left\{ e^{-uT_\Omega(\mathbf{y}, \mathbf{r}(\cdot))} \right\} = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j + \frac{1}{u}} \right) \int_\Omega \phi_j(\mathbf{r})\, d\mathbf{r}\, \frac{1}{2\pi} \int_\Omega \frac{\phi_j(\boldsymbol{\rho})\, d\boldsymbol{\rho}}{|\boldsymbol{\rho} - \mathbf{y}|}$$

1. We note that $0 \leq 1 - h(\mathbf{y}, u) \leq 1$
2. The function $1 - h(\mathbf{y}, u)$ is non-decreasing in u: $1 - h(\mathbf{y}, u_1) \leq 1 - h(\mathbf{y}, u_2)$ if $u_1 < u_2$
3. This is true due to the following
   3.1 $0 \leq e^{-uT_\Omega(\mathbf{y}, \mathbf{r}(\cdot))} \leq 1$ and
   3.2

$$\lim_{u \nearrow \infty} e^{-uT_\Omega(\mathbf{y}, \mathbf{r}(\cdot))} = \begin{cases} 0, & T_\Omega > 0 \\ 1, & T_\Omega = 0 \end{cases}$$

Probabilistic Potential Theory

- From the previous results and the bounded convergence theorem we have

$$\mathcal{U}(\mathbf{y}) = \lim_{u \nearrow \infty} (1 - h(\mathbf{y}, u)) = P\left\{T_\Omega(\mathbf{y}, \mathbf{r}(\cdot)) > 0\right\}$$

  and hence also

$$\mathcal{U}(\mathbf{y}) = \lim_{u \nearrow \infty} \sum_{j=1}^{\infty} \left(\frac{1}{\frac{1}{u} + \lambda_j} \right) \int_\Omega \phi_j(\mathbf{r})\, d\mathbf{r}\, \frac{1}{2\pi} \int_\Omega \frac{\phi_j(\boldsymbol{\rho})\, d\boldsymbol{\rho}}{|\boldsymbol{\rho} - \mathbf{y}|}$$

  and this holds $\forall \mathbf{y} \in \mathbb{R}^3$

Case 1. Let $\mathbf{y} \in \Omega^o$ (the interior), clearly the continuity of $\mathbf{r}(\cdot)$ immediately implies

$$\mathcal{U}(\mathbf{y}) = P\left\{T_\Omega(\mathbf{y}, \mathbf{r}(\cdot)) > 0\right\} = 1$$

Remark: with $\mathbf{y} \in \Omega^o$ we have $\mathcal{U}(\mathbf{y}) = 1$ and so we have the following summability result

$$1 = \lim_{u \nearrow \infty} \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\lambda_j + \frac{1}{u}} \right) \int_\Omega \phi_j(\mathbf{r})\, d\mathbf{r}\, \frac{1}{2\pi} \int_\Omega \frac{\phi_j(\boldsymbol{\rho})\, d\boldsymbol{\rho}}{|\boldsymbol{\rho} - \mathbf{y}|}$$

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Case 2. Let $\mathbf{y} \notin \Omega$, we already know that $1 - h(\mathbf{y}, u)$ is harmonic in $\mathbf{y}$, and it is nondecreasing in u, and the previous limit in u exists and equals $P\{T_\Omega(\mathbf{y}, \mathbf{r}(\cdot)) > 0\}$, thus by Harnack's theorem, $\mathcal{U}(\mathbf{y})$ is harmonic with $\mathbf{y} \notin \Omega$. Assume that $\Omega \subset S(0, a)$, then

$$P\{T_\Omega(\mathbf{y}, \mathbf{r}(\cdot)) > 0\} \leq P\left\{T_{S(0,a)}(\mathbf{y}, \mathbf{r}(\cdot)) > 0\right\}$$

From the last problem this means

$$P\{T_\Omega(\mathbf{y}, \mathbf{r}(\cdot)) > 0\} \leq \frac{a}{|\mathbf{y}|}, \quad \mathbf{y} \notin S(0, a)$$

and so $\lim_{|\mathbf{y}| \to \infty} \mathcal{U}(\mathbf{y}) = 0$

Case 3. Let $\mathbf{y}_o \in \partial\Omega$, and assume that it is regular in the sense of Poincaré: $\exists$ a sphere $S(\mathbf{y}_*, \epsilon)$ lying completely in Ω so that $\mathbf{y}_o \in S(\mathbf{y}_*, \epsilon)$ Consider now $\mathbf{y} \notin \Omega$

$$\mathcal{U}(\mathbf{y}) = P\{T_\Omega(\mathbf{y}, \mathbf{r}(\cdot)) > 0\} \geq P\left\{T_{S(0,a)}(\mathbf{y}, \mathbf{r}(\cdot)) > 0\right\} = \frac{\epsilon}{|\mathbf{y} - \mathbf{y}_*|}$$

As $\mathbf{y} \to \mathbf{y}_o$ with $\mathbf{y} \notin \Omega$ we have $\frac{\epsilon}{|\mathbf{y}-\mathbf{y}_*|} \to \frac{\epsilon}{|\mathbf{y}_o-\mathbf{y}_*|}$, and since $\mathcal{U}(\mathbf{y}) \leq 1$ we have finally that

$$\lim_{\mathbf{y} \to \mathbf{y}_o} \mathcal{U}(\mathbf{y}) = 1$$

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- Thus if Ω is a closed and bounded region, each point on the boundary that is regular in the Poincaré sense has $\mathcal{U}(\mathbf{y})$ as the capacitory potential of Ω
- Recall that

$$\mathcal{U}(\mathbf{y}) = \lim_{\delta \to 0} \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j + \delta} \right) \int_{\Omega} \phi_j(\mathbf{r})\, d\mathbf{r}\, \frac{1}{2\pi} \int_{\Omega} \frac{\phi_j(\boldsymbol{\rho})\, d\boldsymbol{\rho}}{|\boldsymbol{\rho} - \mathbf{y}|}$$

- We note that this implies that

$$\lim_{|\mathbf{y}| \to \infty} |\mathbf{y}|(1 - h(|\mathbf{y}|, u)) = \frac{1}{2\pi} \int_{\Omega} u h(\boldsymbol{\rho}, u)\, d\boldsymbol{\rho}$$

- Again assume that $\Omega \in S(0, a)$, then $h(\mathbf{y}, u) = E\left\{e^{-uT_{\Omega}}\right\} \geq \left\{e^{-uT_{S(0,a)}}\right\}$, there for $\mathbf{y} \notin S(0, a)$ we have $h(\mathbf{y}, u) \geq 1 - \frac{a}{|\mathbf{y}|}$ or $1 - h(\mathbf{y}, u) \leq \frac{a}{|\mathbf{y}|}$ and so

$$\frac{u}{2\pi} \int_{\Omega} h(\boldsymbol{\rho}, u)\, d\boldsymbol{\rho} \leq a$$