Chapter II: Brownian Motion (Wiener Measure)

Let Ω be cont. functions $\beta(s)$ $s\in[0,1]$ and $\beta(0)=0$.

Ask $P\{\int_0^1 \beta^2(s)\,ds \le \alpha\} = ?$

Consider $(\Omega, \mathcal{B}, P)$ some idea of what we have been talking about.

Let $X_1, X_2, \dots$ be indep. ident. distr. r.v.'s with $\mu=0$ and $\sigma^2=1$ and C.L.T.

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \le \alpha\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}}\,du \quad \text{in } 1944$$

Erdös & Kac proved $\lim_{n\to\infty} P\left\{\max\left(\frac{S_1}{\sqrt{n}}, \frac{S_2}{\sqrt{n}}, \dots, \frac{S_n}{\sqrt{n}}\right) \le \alpha\right\} = \sigma_1(\alpha)$

found this.

$\sigma_1(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\alpha} e^{-\frac{u^2}{2}}\,du$ and under the same hypotheses on the r.v.'s they proved

$$\lim_{n\to\infty} P\left\{\frac{S_1^2+S_2^2+\cdots+S_n^2}{n^2} \le \alpha\right\} = \sigma_2(\alpha) \quad \text{and}$$

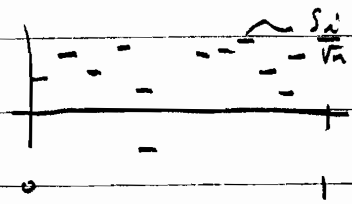
$$\lim_{n\to\infty} P\left\{\frac{S_1+S_2+\cdots+S_n}{n^{3/2}} \le \alpha\right\} = \sigma_3(\alpha) \quad \text{and}$$

Let N_n = the # of partial sums out of $S_1, S_2, \dots, S_n$ which are strictly positive then

$$\lim_{n\to\infty} P\left\{\frac{N_n}{n} \le \alpha\right\} = \begin{cases} 0 & \alpha \le 0 \\ \frac{2}{\pi}\arcsin\sqrt{\alpha} & 0 \le \alpha \le 1 \\ 1 & \alpha \ge 1 \end{cases}$$

this is the arcsine law

Again let the X_i's be i.i.d. with $\mu=0$, $\sigma^2=1$. For each $n \in J$ and $t \in [0,1]$ define $x^{(n)}(t) = \begin{cases} \frac{S_1}{\sqrt{n}} & t=0 \\ \frac{S_i}{\sqrt{n}} & \frac{i-1}{n} < t \le \frac{i}{n} \quad i=1,2,\dots,n \end{cases}$



Let F: $\mathcal{R}$ (space of Riemann integrable function on $[0,1]$) $\to \mathbb{R}$ with weak hypotheses. Then

Theorem: $\lim_{n\to\infty} P\{F[x^{(n)}(\cdot)] \le \alpha\} = P_W\{F[\beta] \le \alpha\}$

connected with — the Wiener measure

Examples: (1) $F[\beta] = \int_0^1 \beta^2(s)\,ds$ the by theorem

$$\lim_{n\to\infty} P\left[\left(\sum_{i=1}^{n} \frac{S_i^2}{n^2}\right) \le \alpha\right] = P_W\left\{\int_0^1 \beta^2(s)\,ds \le \alpha\right\}$$

2) $F[\beta] = \beta(1)$ so $\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \le \alpha\right\} = P_W\{\beta(1) \le \alpha\}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}}\,du$$

3) Let $F[\beta] = \int_0^1 \frac{1+\text{signum}\,\beta(s)}{2}\,ds$ $\qquad \text{signum}\,x = \begin{cases} 1 & x>0 \\ -1 & x \le 0 \end{cases}$

so $\lim_{n\to\infty} P\left\{\frac{U}{n} \le \alpha\right\} = P_W\left\{\int_0^1 \frac{1+\text{signum}\,\beta(s)}{2}\,ds \le \alpha\right\}$.

Formal: For any integer n and any choice of $0 < \tau_1 < \tau_2 < \cdots < \tau_n \le 1$ and any Lebesgue measurable set in $\mathbb{R}^n$ we define an "interval" $C_0[0,1]$ as follows

$$I = I(n;\tau_1,\tau_2,\dots,\tau_n;E) = \{\beta(\cdot)\in C_0[0,1] : (\beta(\tau_1),\beta(\tau_2)\dots,\beta(\tau_n))\in E\}$$

Let $\mathcal{I}$ be the class of intervals, i.e. for all n, all choices of $\tau_1,\tau_2,\dots,\tau_n$ and all Lebesgue meas. sets $E\subset\mathbb{R}^n$. We we see that $\mathcal{I}$ is an algebra of sets in $C_0[0,1]$. Let $\mathcal{B}$ be the smallest σ-algebra generated by $\mathcal{I}$, this is the class of Wiener measurable sets.

Given an interval I we define $m(I) = \dfrac{1}{\sqrt{(2\pi)^n\tau_1(\tau_2-\tau_1)\cdots(\tau_n-\tau_{n-1})}}\displaystyle\iint\!\!\cdots\!\!\int_E e^{-\frac{u_1^2}{2\tau_1}-\frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)}-\cdots-\frac{(u_n-u_{n-1})^2}{2(\tau_n-\tau_{n-1})}}du_1\dots du_n$

Very strange.

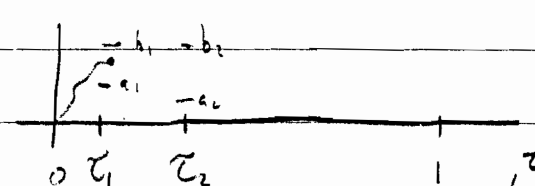
Let $A_{ij} = \min(\tau_i,\tau_j)$ an $n\times n$ matrix $\quad n=3 \quad \tau_1<\tau_2<\tau_3$

$$A = \begin{pmatrix}\tau_1 & \tau_1 & \tau_1\\ \tau_1 & \tau_2 & \tau_2\\ \tau_1 & \tau_2 & \tau_3\end{pmatrix}$$

$$m(I) = \frac{1}{\sqrt{(2\pi)^2\det A}}\iint\!\cdots\!\int e^{-U^TA^{-1}U}du_1\cdots du_n$$

$$U = \begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}$$

Consider



one component of a Brownian Motion.

$$P\{a_1\le\beta(\tau_1)\le b_1\} = \frac{1}{\sqrt{2\pi\tau_1}}\int_{a_1}^{b_1}e^{-\frac{u^2}{2\tau_1}}du$$ assuming Gaussian

$$P\{a_1\le\beta(\tau_1)\le b_1 \text{ and } a_2\le\beta(\tau_2)\le b_2\} = \frac{1}{\sqrt{(2\pi)^2\tau_1(\tau_2-\tau_1)}}\int_{a_2}^{b_2}\int_{a_1}^{b_1} e^{-\frac{u_1^2}{\tau_1}-\frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)}}du_1 d$$

Theorem: If $I = \bigcup_{j=1}^{\infty} I_j$ where $I_j \cap I_k = \phi$ $\forall i \neq k$ and $I \in \mathcal{C}$, $I_j \in \mathcal{C}$ $j = 1, 2, 3, \cdots$ then $m(I) = \sum_{j=1}^{\infty} m(I_j)$.

Let n = dimension, τ_j are the defining points & E the defining set.
Let I_1, I_2 be intervals

$$I_1(n_1; \tau_1^{(1)}, \tau_2^{(1)}, \ldots, \tau_{n_1}^{(1)}; E^{(1)}) \qquad \text{suppose } n_2 > n_1$$
$$I_2(n_2; \tau_1^{(2)}, \tau_2^{(2)}, \ldots, \tau_{n_2}^{(2)}; E^{(2)})$$

Suppose $I_1 = \{\beta(\cdot) \in C_0[0,1] \mid \beta(\tfrac{1}{2}) \leq 2\}$ also we can write
$I_1 = \{\beta(\cdot) \in C_0[0,1] \mid \beta(\tfrac{1}{2}) \leq 2 \text{ \& } -\infty \leq \beta(\tfrac{3}{4}) \leq \infty\}$
using trivial restrictions. So we make I_1 & I_2 such that they have the same dimension & same defining points and so $\mathcal{C}$ is an algebra since $C_0[0,1] \in \mathcal{C}$ and $I_1 \cup I_2 \in \mathcal{C}$ complimen-
tc. Sets on $\mathcal{L}^2$ topology are measurable if originally measurable
Sets " uniform " " " " " "

We will also discover that 1) almost all paths are non-differentia[ble]
at every point and 2) a.e path will satisfy a Hölder (Lip-α) conditi[on]
of order α where $\alpha < \frac{1}{2}$ $\ni$ $|\beta(s) - \beta(\tau)| \leq h|s-\tau|^{\alpha}$.

Let $E \subset \mathbb{R}^n$ $\quad 0<\tau_1<\tau_2<\cdots<\tau_n<1$,

$$I = \{\beta \in C_0[0,1] : (\beta(\tau_1), \cdots, \beta(\tau_n)) \in E\}$$

E: measurable set in $\mathbb{R}^n$

define $$m(I) = \frac{1}{\sqrt{(2\pi)^n \tau_1(\tau_2-\tau_1)\cdots(\tau_n-\tau_{n-1})}} \int\int_E\cdots\int e^{-\frac{u_1^2}{2\tau_1} - \cdots - \frac{(u_n-u_{n-1})^2}{2(\tau_n-\tau_{n-1})}} du_1 \cdots du_n$$

$$= \int\int\int\cdots\int p(\tau_1, 0, u_1)\, p(\tau_2-\tau_1, u_1, u_2)\cdots p(\tau_n-\tau_{n-1}, u_n, u_{n-1})\, du_1 \cdots du_n$$

where $p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$, notice this solves

$$\varphi_t = \frac{1}{2}\varphi_{yy}$$
$$\varphi(y,0) = \delta(y-x)$$

It is obvious that this finitely additive

$I = I_1 \cup I_2$, $I_1 \cap I_2 = \phi$ since integrals are additive set functions.

Theorem 1: Let $a>0$, $0<\gamma<\frac{1}{2}$, and define

$$A_{a,\gamma} = \{\beta \in C_0[0,1] \,/\, |\beta(\tau_2)-\beta(\tau_1)| \le a|\tau_2-\tau_1|^\gamma \ \forall \tau_2, \tau_1 \in [0,1]\}$$

For any interval $I \subset C_0[0,1] \ni I \cap A_{a,\gamma} = \phi$

$m(I) < K a^{-\frac{4}{1-2\gamma}}$ $\quad K$ is indep. of a.

But $A_{a,\gamma}$ is a compact set in $C_0[0,1]$ and eventually one can prove that almost all $\beta \in C_0[0,1]$ satisfy some Hölder cond.

Theorem 2: m is countably additive on $\mathcal{A}$, i.e. if $I_n \in \mathcal{A}$ which are disjoint $I_j \cap I_k = \phi$, $j \ne k$ then if

$$I = \bigcup_{n=1}^{\infty} I_n \in \mathcal{A} \text{ then } m(I) = \sum_{n=1}^{\infty} m(I_n).$$

Proof: Let $J_n = I - \bigcup_{j=1}^{n} I_j$. Clearly $J_n \supset J_{n-1}$ and $\lim_{n\to\infty} J_n = \bigcap_{n=1}^{\infty} I_n = \emptyset$. From finite additivity we have

$$m(I) = \sum_{j=1}^{n} m(I_j) + m(J_n)$$

so it suffices to show that

$$\lim_{n\to\infty} m(J_n) = 0.$$

Let $\tau_1^{(n)}, \tau_2^{(n)}, \dots, \tau_{s_n}^{(n)}$ and E_n a Lebesgue measurable set $\subset \mathbb{R}^{s_n}$ be the definition of J_n. defining stuff

Clearly $\exists\, E_n^*$ closed $\subset \mathbb{R}^{s_n} \ni E_n^* \subset E_n \ni$ the interval K_n whose defining points are also $\tau_1^{(n)}, \dots, \tau_{s_n}^{(n)}$ but whose defining set is E_n^* with $m(E_n - E_n^*) < \frac{\varepsilon}{2^{n+1}}$.

Let $L_n = \bigcap_{j=1}^{n} K_j$ then $L_n \subset K_n \subset J_n$ and

$$m(J_n) = m(J_n - L_n) + m(L_n)$$

since

$$J_n - L_n = J_n - \bigcap_{j=1}^{n} K_j \subset \bigcup_{j=1}^{n} (J_j - K_j)$$

$$\therefore\ m(J_n - L_n) \le \sum_{j=1}^{n} m(J_j - K_j) < \sum_{j=1}^{n} \frac{\varepsilon}{2^{j+1}} < \frac{\varepsilon}{2}$$

So for all $n > N$, $m(J_n) < m(L_n) + \frac{\varepsilon}{2}$. Want to show this is small.

From Theorem 1 for any $\gamma \in (0, \frac{1}{2})$ $\exists$ $a > 0$ so large $\ni$ if $I \cap A_{a,\gamma}$ then

$m(I) < \frac{\varepsilon}{2}$. Hence it will suffice for such an $a > 0$ to find an N_0 so large that $n > N_0$ implies

$$L_n \cap A_{a,\gamma} = \emptyset.$$

Let $M_n = L_n \cap A_{n,\gamma}$ since M_n is decreasing by L_n decreasing and

$\lim_{n\to\infty} M_n \subset \lim_{n\to\infty} J_n = \phi$. We must that $\exists N_0 \ni \forall n > N_0 \; M_n = \phi$.

Assume the contrary, i.e $\exists$ a sequence $\beta_{nk} \in C_0[0,1] \ni \beta_{nk} \in M_{nk}$ $k = 1, 2, 3, \ldots$

For any n_{k_0} $\beta_{nk} \in M_{n_{k_0}}$ $\forall n_k \geq n_{k_0}$. Consider this infinite set of functions belonging to $M_{n_{k_0}} = L_{n_{k_0}} \cap A_{a,\gamma}$. Since each of these functions belong to $A_{a,\gamma}$ they constitute a uniformly bounded equicontinuous family, by Ascoli's theorem $\exists$ a subsequence converging to a function $\beta_0 \in A_{a,\gamma}$ more over since each β_{nk} with $n_k > n_{k_0}$ is in $L_{n_{k_0}}$ which is an interval whose defining set is closed. Therefore we have $\beta_0 \in L_{n_{k_0}}$ $\therefore$ $\beta_0 \in M_{n_{k_0}}$, so $\beta_0 \in M_{n_{k_0}}$ with n_{k_0} arbitrary so $\beta_0 \in M_{n_k}$ $\forall n_k$ which contradicts $\lim_{n\to\infty} M_n = \phi$. Which proves the theorem (Theorem 2).

Lemma 1: For any positive integer n, let $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{2^n} = 1$

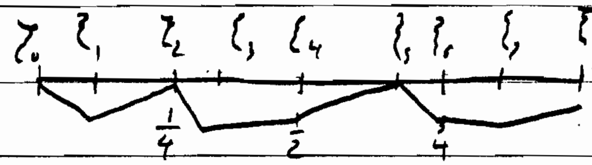
where $\xi_j = \frac{j}{2^n}$ $j = 0, 2, 4, 6, \ldots, 2^n$. Let $\beta \in C_0[0,1]$ and linear between ξ_{i-1} and ξ_i, $i = 1, 2, \cdots, 2^n$. Suppose $\exists\, a > 0$ and $0 < \gamma < \frac{1}{2}$ $\ni \forall k = 1, 2, \cdots, n$ and all $l = 1, 2, 3, \cdots 2^{n-k}$

$$|\beta(\xi_{l 2^k}) - \beta(\xi_{(l-1)2^k})| \leq \left(\frac{1 - 2^{-\gamma}}{24}\right) a \,|\xi_{l \cdot 2^k} - \xi_{(l-1)2^k}|^\gamma$$

Prob.

then $\forall\ \tau_2, \tau_1 \in [0,1]$, $|\beta(\tau_2) - \beta(\tau_1)| \le a|\tau_2 - \tau_1|^{\gamma}$.

A picture:



differences over gaps of $1, 2, 8, \cdots 2^n$ are required to prove this.

Lemma 2: Let $0 \le \tau_1 < \tau_2 \le 1$ and $I = \{\beta \in C_0[0,1] \mid |\beta(\tau_2) - \beta(\tau_1)| > a|\tau_2 - \tau_1|^{\gamma}\}$

then $m(I) \le e^{\frac{-a^2}{2(\tau_2-\tau_1)^{1-2\gamma}}}$.

Proof: Note that I is an interval if we define $E \subset \mathbb{R}_2 = \{(u_1, u_2) \in \mathbb{R}^2 \mid |u_2 - u_1| > a|\tau_1 - \tau_1|^{\gamma}\}$ then

$$I = \{\beta \in C_0[0,1] : (\beta(\tau_1), \beta(\tau_2)) \in E\}.$$

But
$$m(I) = \frac{1}{\sqrt{(2\pi)^2 \tau_1 (\tau_2 - \tau_1)}} \iint_E e^{-\frac{u_1^2}{2\tau_1}} e^{-\frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)}} du_1\, du_2$$

$$= \frac{1}{\sqrt{(2\pi)^2 \tau_1(\tau_2-\tau_1)}} \left(\int_{-\infty}^{\infty} e^{-v_1^2/2\tau_1} dv_1\right)\left(\int_{|v_2|>b} e^{-v_2^2/2(\tau_2-\tau_1)} dv_2\right)$$

$b = a|\tau_2 - \tau_1|^{\gamma}$

$$= \frac{2}{\sqrt{2\pi(\tau_2-\tau_1)}} \int_0^{\infty} e^{-\frac{(u+b)^2}{2(\tau_2-\tau_1)}} du$$ ← throw out cross term & integrate u^2

$$\le e^{-b^2/2(\tau_2-\tau_1)} = e^{\frac{-a^2}{2(\tau_2-\tau_1)^{1-2\gamma}}}.$$

Theorem 1: Proof: Let $\tau_1, \tau_2, \cdots, \tau_s$ and $E^* \in \mathbb{R}^s$ be the definiti of I, i.e.

$$I = \{\beta \in C_0[0,1] \mid (\beta(\tau_1), \beta(\tau_2), \cdots, \beta(\tau_s)) \in E^*\}.$$

Choose n so large that $\frac{1}{2^{n-1}} < \min(\tau_1, \tau_2-\tau_1, \dots, \tau_s-\tau_{s-1})$ and if $\tau_s \neq 1$ $\frac{1}{2^{n-1}} < (1-\tau_s)$. For such an n let

$$\xi_j = \frac{j}{2^n} \quad \text{for } j=0,2,4,6,\dots,2^n.$$

Clearly between any two successive ξ_j's $j=0,2,4,\dots,2^n$ there is at most one τ point. If τ_k falls between ξ_{2j} and ξ_{2j+2} define $\xi_{2j+1} = \tau_k$. If no τ falls between define $\xi_{2j+1} = \frac{\xi_{2j}+\xi_{2j+2}}{2}$.

Thus $S = \{\xi_0, \xi_1, \dots, \xi_{2^n}\} \supset (\tau_1, \tau_2, \dots, \tau_s)$ and by imposing trivial restrictions at appropriate points I can be defined in terms of S and a Lebesgue measurable set in $\mathbb{R}^{2^n}$. For any $\beta \in I$ define

$$\beta^* = \begin{cases} \beta(\tau) & \text{for } \tau \in S \\ & \text{linear inbetween the points of } S \end{cases}$$

Thus $\beta^* \in I$ $\therefore$ $\beta^* \in A_{a,\gamma}$. Hence from lemma 1

$$\exists\, k=0,1,2,\dots,n \text{ and } l=1,2,\dots,2^{n-k} \ni$$

$$|\beta^*(\xi_{l\cdot 2^k}) - \beta^*(\xi_{(l-1)2^k})| > \left(\frac{1-2^{-\gamma}}{24}\right) a \,|\xi_{l\cdot 2^k} - \xi_{(l-1)2^k}|^{\gamma}.$$

by lemma 1.

Since $\beta^*(\tau) = \beta(\tau)$ at all $\tau \in S$ the last inequality holds for β also. Let $J_{k,l} = \{\beta \in C_0[0,1] \mid |\beta(\xi_{2^k l}) - \beta(\xi_{2^k(l-1)})| > \left(\frac{1-2^{-\gamma}}{24}\right) a \,|\xi_{l 2^k} - \xi_{(l-1)2^k}|^{\gamma}\}$

We have just shown that $I \subset \bigcup_{k=0}^{n} \bigcup_{l=1}^{2^{n-k}} J_{k,l}$ so

by finite subadditivity $\quad m(I) \leq \sum_{k=0}^{n} \sum_{l=1}^{2^{n-k}} m(J_{k,l})$ but letting $b = \left(\frac{1-2^{-\gamma}}{24}\right) a$ we have from lemma 2

$$m(I) \le \sum_{k=0}^{n} \sum_{\ell=1}^{2^{n-k}} e^{-\frac{b^2}{2(\xi_{\ell 2^k} - \xi_{(\ell-1)2^k})^{1-2\gamma}}}.$$

For $k=0$ $\xi_\ell - \xi_{\ell-1} < \frac{2}{2^n}$

$k \ge 1$ $\xi_{\ell 2^k} - \xi_{(\ell-1)2^k} = \frac{2^k}{2^n}$ $\therefore$ $m(I) < 2^n e^{-\frac{b^2}{2}\cdot 2^{(n-1)(1-2\gamma)}} + \sum_{k=1}^{n} 2^{n-k} e^{-\frac{b^2}{2} 2^{(n-k)(1-2\gamma)}}$

$$= 2\cdot 2^{n-1} e^{-\frac{b^2}{2} 2^{(n-1)(1-2\gamma)}} + \sum_{k=1}^{n} 2^{k-1} e^{-\frac{b^2}{2} 2^{(k-1)(1-2\gamma)}}$$

So $m(I) < 3\sum_{k=1}^{\infty} 2^{k-1} e^{-\frac{b^2}{2}\cdot 2^{(k-1)(1-2\gamma)}} = 3\sum_{k=1}^{\infty} 2^{-k}\left(2^{2k} e^{-\frac{b^2}{2} 2^{k(1-2\gamma)}}\right)$

Let $\phi(u) = u^2 e^{-\frac{b^2}{2}u^\lambda}$ $u \ge 0$, $\lambda > 0$ its maximum is at $u = \left(\frac{4}{\lambda^2 b}\right)^{\frac{1}{\lambda}}$ with maximum $\left(\frac{\lambda b^2 \ell}{4}\right)^{-\frac{2}{\lambda}}$

So replacing with the maximum $< 3\left(\frac{\lambda b^2 \ell}{4}\right)^{-\frac{2}{\lambda}} \sum_{k=0}^{\infty} 2^{-k}$ $\lambda = 1-2\gamma$

$$= 6\left[\frac{(1-2\gamma)(1-2^{-\gamma})^2 e}{4(24)^2}\right]^{-\frac{2}{1-2\gamma}} a^{-\frac{4}{1-2\gamma}}.$$

We get the next theorem on xerox next time. The lemma

Suppose $F: C_0[0,t] \to \mathbb{R}$ and it is measurable, i.e. $\{\beta \in C_0[0,t] \mid F[\beta] \le \alpha\}$ is measurable, $\forall \alpha$

We can consider $E\{F\} = E\{F[\beta(\cdot)]\} = \int F(\beta(\cdot))\, \delta_w \beta$.

Prob.

Consider $C_x[0,t]$ $\quad x\,\overline{\vert\!\!\frown}$ changes only the first term,

But $P\{\beta(0)=x,\ \beta(t)=A\} = \frac{1}{\sqrt{2\pi t}}\int_A e^{-\frac{(y-x)^2}{2t}}\,dy$ but

$p(t,x,y)$ satisfies $\quad u_t = \frac{1}{2}u_{xx}$

$u(x,0)=\delta(x)$, i.e. $\lim_{t\to 0}\int_{-c}^{c} p(t,x,y)\,g(y)\,dy = g(x)$.

$p(t,x,y)$ is the fundamental solution of the heat equation. Now

$$E\{\beta(\tau)\} \quad ; \quad E\{g(\beta(\tau_1),\beta(\tau_2)\cdots,\beta(\tau_n))\}$$

$$= \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} u\, e^{-\frac{u^2}{2\tau}}\,du \qquad = \frac{1}{\sqrt{(2\pi)^n\tau_1\cdots(\tau_n-\tau_{n-1})}}\int\cdots\int g(u_1,\dots,u_n)\,e^{-\frac{u_1^2}{2\tau_1}\cdots-\frac{(u_n-u_{n-1})^2}{2(\tau_n-\tau_{n-1})}}\,du_1\cdots du_n$$

But $P\{\beta(\tau)\le\alpha\} = m(I=\{\beta\in C_0[0,t]\,|\,\beta(\tau)\le\alpha\}) = \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\alpha} e^{-\frac{u^2}{2\tau}}\,d$

Let us use δ function notation. Let g be Borel measurable, then

$$E\{g(\beta(\tau))\} = \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} g(u)\,e^{-\frac{u^2}{2\tau}}\,du \quad, \text{let } g(u)=\delta(u-x)$$

$$E\{\delta(\beta(t)-x)\} = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\delta(u-x)\,e^{-\frac{u^2}{2t}}\,du = \frac{1}{\sqrt{2\pi t}}\,e^{-\frac{x^2}{2t}}$$

Consider $\quad u_t = \frac{1}{2}u_{xx}$ $\qquad$ so $u(x,t) = E\{\delta(\beta(t)-x)\} = \frac{1}{\sqrt{2\pi t}}\,e^{-\frac{x^2}{2t}}$

$u(x,0)=\delta(x)$

Consider $\quad u_t = \frac{1}{2}u_{xx} - V(x)\,u \qquad u(x,0)=\delta(x) \qquad V(x)\ge 0$ continuou

But we can write $u(x,t) = E\left\{ e^{-\int_0^t V(\beta(s))ds} \delta(\beta(t)-x) \right\}$.
This is the Feynman-Kac formula.

<u>Example</u>: $u_t = \frac{1}{2} u_{xx} - \frac{x^2}{2} u \qquad u(x,0) = \delta(x)$, $V = \frac{x^2}{2}$

$$u(x,t) = E\left\{ e^{-\frac{1}{2}\int_0^t \beta^2(s)ds} \delta(\beta(t)-x) \right\}.$$

<u>Theorem:</u> Sufficiency 1922 Lindeberg; Necessity 1937 Feller.
Let $X_1, X_2, \ldots$ be indep. r.v.'s with means μ_k and variance σ_k^2.
Let $S_n = \sum_{j=1} X_j$. A necessary and sufficient condition that

$$\lim_{n\to\infty} P\left\{\frac{S_n - E(S_n)}{\sigma(S_n)} \le x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du$$ and that

the r.v.: $\frac{X_k-\mu_k}{\sigma(S_n)}$ be infinitesimal is that $\forall\, \varepsilon > 0$ the
<u>Lindeberg Condition</u>: $\lim_{n\to\infty} \frac{1}{\sigma^2(S_n)} \sum_{k=1} \int_{|x|>\varepsilon\sigma(S_n)} x^2\, dF_k(x+\mu_k) = 0.$

What does the Lindeberg Condition mean? Consider

$$P\left\{\max_{1\le k\le n} |X_k - \mu_k| > \varepsilon\sigma(S_n)\right\} = P\left\{|X_k-\mu_k| > \varepsilon\sigma(S_n) \text{ for at least one } 1\le k\le n\right\}$$

$$\le \sum_{k=1}^{n} P\{|X_n - \mu_n| > \varepsilon\sigma(S_n)\} = \sum_{k=1}^{n} \int_{|x-\mu_k|>\varepsilon\sigma(S_n)} dF_k(x)$$

$$\le \frac{1}{\varepsilon^2\sigma^2(S_n)} \sum_{k=1}^{n} \int_{|x-\mu_k|>\varepsilon\sigma(S_n)} (x-\mu_k)^2\, dF_k(x) = \frac{1}{\varepsilon^2}\left[\frac{1}{\sigma^2(S_n)} \sum_{k=1} \int_{|x|>\varepsilon\sigma(S_n)} x^2\, dF_k(x-\mu_k)\right].$$

This is much stronger than infinitesimal, the variables are really uniformly really small.

<u>Theorem</u>: Let X_{nk} be a sequence of finite sequences of independent random variables with means μ_{nk} & variances σ_{nk}^2. A n-s condition that $S_n = \sum_{j=1}^{k_n} X_{nj} - A_n$ where $\{A_n\}$ is an appropriately selected sequence of constants, converge to $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int^{x} e^{-u^2/2}\,du$, that $\sigma^2(S_n) \to 1$, and that the variables be infinitesimal is that $\forall\, \varepsilon > 0$

1) $\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\ge\varepsilon} x^2\, dF_k(x+\mu_k) = 0$

2) $\lim \sum^{k_n} \int x^2\, dF_k(x+\mu_k) = 1$

Proof: Sufficiency: Consider $\sup_{1\le k\le k_n} P\{|X_{nk}-\mu_{nk}|\ge\varepsilon\} = \sup_{1\le k\le k_n}\int_{|x|\ge\varepsilon} dF_{nk}(x+\mu_{nk})$

$\le \frac{1}{\varepsilon^2}\sup_{1\le k\le k_n}\int_{|x|\ge\varepsilon} x^2\,dF_{nk}(x_{nk}+\mu_{nk}) \le \frac{1}{\varepsilon^2}\sum_{k=1}^{k_n}\int_{|x|\ge\varepsilon} x^2\,dF_{nk}(x_{nk}+\mu_{nk})$

so they are infinitesimal. Moreover $\sigma^2(S_n)\to 1$ by 1) and 2) $\therefore \sigma^2(S_n)$ is uniformly bounded and conditions α & β of Bawly's theorem are satisfied. Therefore by Bawly's theorem looking at the accompanying laws. A sufficient condition for convergence $\Phi(x)$ of the d.f.'s is that

$$K_n(x) = \sum_{k=1}^{k_n}\int_{-\infty}^{x} u^2\,dF_{nk}(u+\mu_{nk}) = \begin{cases}0 & x<0\\ 1 & x>0\end{cases}$$

and $K_n(+\infty)\to_{k_n} K(+\infty)$ since $\sigma^2(S_n)\to 1$ since we can choose $A_n=\sum_{k=1}^{k_n}\mu_{nk}$ we get $\gamma_n\to\gamma$ with $\gamma_n=\sum_{k=1}^{k_n}\mu_{nk}-A_{nk}$.

Necessity Trivial

Corollary: Suppose $\sigma^2(S_n)=1$ $\forall n$ then conditions 1) & 2) reduce to $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|\ge\varepsilon} x^2\,dF_{nk}(x-\mu_{nk})=0$ and Moreover if

$\mu_{nk}=0$ $\forall n$, $k=1,2,\dots,k_n$ then this last one is

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|\ge\varepsilon} x^2\,dF_{nk}(x_{nk}+\mu_{nk}),$$ to see that Lindeberg

theorem follows from this corollary let $X_{nk}=\frac{X_k-\mu_k}{\sigma(S_n)}$ and $S_n^{*}=X_{n1}+\dots+X_{nk_n}$ and $S_n^{*}=\frac{S_n-E(S_n)}{\sigma(S_n)}$ and

$F_{nk}(x)=P\{X_{nk}\le x\}=P\left\{\frac{X_k-\mu_k}{\sigma(S_n)}\le x\right\}=F_k(x\sigma(S_n)+\mu_k)$ and the condition to be checked is

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|\ge\varepsilon} x^2\,dF_k(x\sigma(S_n)+\mu_k)=0$$ which is

$$\lim_{n\to\infty} \frac{1}{\sigma^2(S_n)} \sum_{k=1}^{k_n} \int_{|x|\geq \varepsilon\sigma(S_n)} x^2 \, dF_{nk}(x+\mu_{nk}) = 0$$

<u>Theorem:</u> Let $\{X_{nk}\}$ be a seq. of fin. seqs. of ind. infinitesimal r.v.'s with means μ_{nk} & variance σ_{nk}^2. A n&s condition that the d.f^{n}s. of $S_n = X_{n1} + \cdots + X_{nk_n} - A_n$ (where the $\{A_n\}$ are properly chosen) converge to the Poisson $P(x) = \sum_{0\leq m\leq x} \frac{e^{-\lambda}\lambda^m}{m!}$ and that $\sigma^2(S_n) \to \lambda$ is that

$\forall \varepsilon > 0$ $$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x-1|\geq\varepsilon} x^2 \, dF_{nk}(x+\mu_{nk}) = 0$$ and

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x-1|<\varepsilon} x^2 \, dF_{nk}(x+\mu_{nk}) = \lambda$$ the $\{A_n\}$ should be selected so that $A_n = \sum_{k=1}^{k_n} \mu_{nk} - \lambda + o(1)$ as $n\to\infty$

We will talk about $(\Omega, \mathcal{B}, P)$ $\sum_{n=1}^{\infty} X_n(\omega)$ converge a.e. for certain cases.

<u>Brownian Motion</u>: $(\Omega, \mathcal{B}, P)$ Wiener measure

$C_x[0,1]$ the open sets

Let $F: \Omega \to \mathbb{R} \ni \{\beta : F[\beta] \leq \alpha\}$ is measurable $\forall \alpha$.
Let $\omega \in (\Omega, \mathcal{B}, P)$ and $X(\omega)$ an r.v. $P\{X \leq x\} = F(x)$ and $g(x)$ is a Borel measurable function in $\mathbb{R}_1$ then

$$E\{g(X)\} = \int_{-\infty}^{\infty} g \, dF(x)$$

and if $X_1, \ldots, X_n$ are any r.v's on $(\Omega, \mathcal{B}, P)$ whose joint density/$\mathbb{R}^n$ is $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ is a Borel measurable function in $\mathbb{R}^n$

$$E\{G(X_1, \ldots, X_n)\} = \iint_{-\infty}^{+\infty}\!\!\cdots\!\int g(x_1, x_2, \ldots, x_n) f(x_1, \ldots, x_n) \, dx_1 dx_2 \cdots dx_n$$

But $P\{\beta(\tau) \le x\} = m\{\beta \in C_0[0,\tau] \mid \beta(\tau) \in E = (-\infty, x]\}$

$$= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{x} e^{-\frac{u^2}{2\tau}}\, du$$ and similarly

$$P\{\beta(\tau_1) \le x_1, \dots, \beta(\tau_n) \le x_n\} = \frac{1}{\sqrt{(2\pi)^n \tau_1 (\dots)(\tau_n - \tau_{n-1})}} \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} e^{\frac{-u_1^2}{2\tau_1} \cdots \frac{(u_n - u_{n-1})^2}{2(\tau_n - \tau_{n-1})}}\, du_1 \cdots du_n$$

Hence $E\{g(\beta(\tau_1), \beta(\tau_2), \dots, \beta(\tau_n)) = \frac{1}{\sqrt{(2\pi)^n |A|}} \iint_{-\infty}^{\infty} e^{-\frac{1}{2} U' A U} g(u_1, \dots, u_n)\, du_1 \dots du_n$
and $A_{ij} = (\min(\tau_i, \tau_j))$.

Suppose $F[\beta] = \int_0^t \beta^2(s)\, ds$ so $E\{\int_0^t \beta^2(s)\, ds\} \overset{\text{by Fubini}}{=} \int_0^t E\{\beta^2(s)\}\, ds$

$$= \int_0^t s\, ds = \frac{t^2}{2}.$$

So $E\{e^{\int_0^t \rho(s)\, ds}\}$ do a little classical analysis. Consider the integral eqn.

$$\rho \int_0^t u(s) \min(\tau, s)\, ds = u(\tau)$$ for what ρ's are there solutions and so let $\rho_0, \rho_1, \rho_2, \dots$

$\int_0^t u_k^2(\tau)\, d\tau = 1$ $\quad u_0(\tau), u_1(\tau), u_2(\tau), \dots$ are normalized.

So $\rho \int_0^\tau s u(s)\, ds + \rho \int_\tau^t \tau u(s)\, ds = u(\tau)$ differentiate w.r.t. τ

$\rho \tau u(\tau) - \rho \tau u(\tau) + \rho \int_\tau^t u(s)\, ds = u'(\tau)$ with and then again with

$-\rho u(\tau) = u''(\tau)$

So $u''(\tau) + \rho u(\tau) = 0$ $\quad u(0) = 0$; $u'(t) = 0$

$$\left.\begin{aligned} \rho_k &= (k + \tfrac{1}{2})^2 \frac{\pi^2}{t^2} \\ u_k(s) &= \sqrt{\tfrac{2}{t}} \sin (k + \tfrac{1}{2}) \frac{\pi s}{t} \end{aligned}\right\} \quad k = 0, 1, 2, \dots$$

So $\min(s, \tau) = \sum_{k=0}^{\infty} \frac{u_k(s) u_k(\tau)}{\rho_k}$ the Spectral Theorem

Let $\alpha_0, \alpha_1, \alpha_2, \cdots$ be indep. r.v.'s each $N(0,1)$. Consider

$$\circledast \quad \sum_{k=0}^{\infty} \frac{\alpha_k(\omega) u_k(\tau)}{\sqrt{\rho_k}} = \beta(\tau) \quad \text{(will prove that this converges for a.e. } \omega)$$

This is a Fourier series which has random coefficients.

$$E\{\beta^2(\tau)\} = E\left\{\sum_{k=0}^{\infty} \frac{\alpha_k(\omega)}{\sqrt{\rho_k}} u_k(\tau) \sum_{l=0}^{\infty} \frac{\alpha_l(\omega) u_l(\tau)}{\sqrt{\rho_l}}\right\} \quad \text{only } k=l \text{ because } N(0,1)$$

exchange summation & integration $= \sum_{l=0}^{\infty} \frac{u_k^2(\tau)}{\rho_k} = \min(\tau, \tau) = \tau$

(This proves $\circledast$ because is $N(0,\tau)$.

So $E\{\beta(\tau)\beta(s)\} = \min(\tau, s)$.

replace with Fourier Series

$$E\left\{e^{\int_0^t \beta(s)\,ds}\right\} = E\left\{e^{\int_0^t \sum_{k=0}^{\infty} \frac{\alpha_k}{\sqrt{\rho_k}} u_k(s)\,ds}\right\}$$

indep. Moment Generating function

$$= E\left\{e^{\sum_{k=0}^{\infty}\int_0^t \frac{\alpha_k}{\sqrt{\rho_k}} u_k(s)\,ds}\right\} = \prod_{k=0}^{\infty} E\left\{e^{\frac{\alpha_k}{\sqrt{\rho_k}}\int_0^t u_k(s)\,ds}\right\}$$

$$= \prod_{k=0}^{\infty} e^{\frac{1}{2\rho_k}\left(\int_0^t u_k(s)\,ds\right)^2} = e^{\frac{1}{2}\int_0^t\int_0^t \sum_{k=0}^{\infty} \frac{u_k(s) u_k(\tau)}{\rho_k}\,ds\,d\tau}$$

$$= e^{\frac{1}{2}\int_0^t\int_0^t \min(s,\tau)\,ds\,d\tau} = e^{t^3/6}$$

$$\text{So } E\left\{e^{-\frac{\lambda^2}{2}\int_0^t \beta^2(s)\,ds}\right\} \overset{\text{Parseval's Theorem}}{=} E\left\{e^{-\frac{\lambda^2}{2}\sum_{k=0}^{\infty}\frac{\alpha_k^2}{\rho_k}}\right\} \overset{\text{ind}}{=}$$

$$\prod_{k=0}^{\infty} E\left\{e^{-\frac{\lambda^2}{2}\frac{\alpha_k^2}{\rho_k}}\right\} = \prod_{k=0}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{2}\frac{\alpha^2}{\rho_k}} e^{-\frac{\alpha^2}{2}}\,d\alpha$$

$$= \prod_{k=0}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{2}\left(1+\frac{\lambda^2}{\rho_k}\right)}\,d\alpha$$

$$= \prod_{k=0}^{\infty}\frac{1}{\sqrt{1+\frac{\lambda^2}{\rho_k}}} = \frac{1}{\sqrt{\prod_{k=0}^{\infty}\left(1+\frac{\lambda^2 t^2}{(k+\frac{1}{2})^2\pi^2}\right)}} = \frac{1}{\sqrt{\cosh(\lambda t)}}$$

by Weierstrass infinite

Let $\lambda = 1$ $E\left\{ e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds} \right\} = \frac{1}{\sqrt{\cosh(t)}}$ how does this behave as $t \to \infty$

$\sim \sqrt{2}\, e^{-\frac{t}{2}}$

And $\lim_{t\to\infty} \frac{1}{t} \log E\left\{ e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds} \right\} = -\frac{1}{2}$

Theorem: If $V(y) \to \infty$ as $|y| \to \infty$ then

$$\lim_{t\to\infty} \frac{1}{t} \log E\left\{ e^{-\int_0^t V(\beta(s))\,ds} \right\} = -\lambda_1$$

where λ_1 is the lowest eigenvalue of $\frac{1}{2}\psi''(y) - V(y)\psi(y) = -\lambda\psi(y)$ the Schroedinger equation. This is small potatoes.

Feynman - Kac : $\begin{cases} u_t = \frac{1}{2} u_{xx} - V(x) u \\ u(x,0) = u_0(x) \end{cases}$ V measurable and bounded below

$$u(x,t) = E_x\left\{ e^{-\int_0^t V(\beta(s))\,ds}\, u_0(\beta(t)) \right\}.$$ Lets play with it

$C_x[0,1]$

Special Cases: 1) $V \equiv 0$ $\quad E_x\{u_0(\beta(t))\} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} u_0(y)\, e^{-\frac{(x-y)^2}{2t}}\,dy$

$= u(x,t)$ as we all know.

2) $V(x) = \frac{x^2}{2}$ so presumably $u(x,t) = E_x\left\{ e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds} (1) \right\}$

$u_0 \equiv 1$

$$= E_0\left\{ e^{-\frac{1}{2}\int_0^t (\beta(s)+x)^2\,ds} \right\}$$

$$= e^{-\frac{x^2 t}{2}} E\left\{ e^{-x\int_0^t \beta(s)\,ds - \frac{1}{2}\int_0^t \beta^2(s)\,ds} \right\}$$

$$= e^{-\frac{x^2 t}{2}} E\left\{ e^{-x\sum_{k=0}^{\infty} \frac{\alpha_k}{\sqrt{\rho_k}} \int_0^t u_k(s)\,ds - \frac{1}{2}\sum_{k=0}^{\infty} \frac{\alpha_k^2}{\rho_k}} \right\} =$$

$$= e^{-\frac{x^2 t}{2}} \prod_{k=0}^{\infty} E\left\{ e^{-x \frac{\alpha_k}{\sqrt{\rho_k}} \int_0^t u_k(\tau)d\tau - \frac{\alpha_k^2}{\rho_k}} \right\}$$

$$= e^{-\frac{x^2 t}{2}} \prod_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x\alpha}{\sqrt{\rho_k}} \int_0^t u_k(s)\,ds} - e^{-\frac{\alpha^2}{2}\left(1+\frac{1}{\rho_k}\right)} d\alpha$$

from completing the square

$$= e^{-\frac{x^2 t}{2}} \frac{1}{\sqrt{\cosh(t)}} e^{\frac{x^2}{2} \underbrace{\int_0^t \int_0^t \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k+1}\,ds\,d\tau}_{-R(s,\tau;-1)}}$$

Where $R(s,\tau;-\lambda^2) + \min(s,\tau) = +\lambda^2 \int_0^t \min(s,\xi) R(\xi,\tau,-\lambda^2)\,ds$

Does $$-\sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k+\lambda^2} + \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k} \overset{?}{=} +\lambda^2 \int_0^t \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\xi)}{\rho_k} \sum_{l=0}^{\infty} \frac{u_l(\xi)u_l(\tau)}{\rho_l+\lambda^2}$$

$$R(s,\tau;-\lambda^2) = \begin{cases} \dfrac{-\cosh\lambda(t-\tau)\sinh(\lambda s)}{\lambda\cosh(\lambda t)} & s\le\tau \\[2ex] \dfrac{-\cosh\lambda(t-s)\sinh(\lambda\tau)}{\lambda\cosh(\lambda t)} & s\ge\tau \end{cases}$$

$0 \le s \le t$

So $$u(x,t) = \frac{1}{\sqrt{\cosh t}}\, e^{-\frac{x^2}{2}\left(t+\int_0^t\int_0^t R(s,\tau;-1)\,ds\,d\tau\right)} = \frac{1}{\sqrt{\cosh t}}\, e^{-x^2 \frac{\tanh t}{2}}.$$

Excercise: $V(x) = \frac{x^2}{2}$; $u_0 = x$, hint

$$u(x,t) = E_x\left\{ e^{-\frac{1}{2}\int_0^t \beta^2(s)\,ds}\, \beta(t) \right\}.$$ Calculate

$$E_x\left\{ e^{\lambda\beta(t) - \frac{1}{2}\int_0^t \beta^2(s)\,ds} \right\} = \tilde{u}(x,t,\lambda) \qquad u(x,t) = \frac{d}{d\lambda}\tilde{u}(x,t,\lambda)\Big|_{\lambda=}$$

Solving for $\psi(x)$ where:

1. $\frac{1}{2}\psi''(x) - (s+\lambda V_\alpha(x))\psi = 0$ $\qquad V_\alpha(x) = \begin{cases} 1 & x \geq \alpha \\ 0 & x < \alpha \end{cases}$

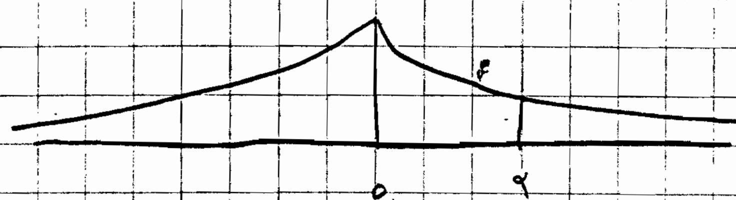
2. $\psi \to 0$ as $x \to \pm\infty$

3. ψ is continuous

4. ψ' is continuous at $x \neq 0$

5. $\psi'(0^-) - \psi'(0^+) = 2$

Case 1: $\alpha > 0$



for $x \geq \alpha$ $\qquad \frac{1}{2}\psi''(x) - (s+\lambda)\psi = 0$

$$\psi''(x) - 2(s+\lambda)\psi = 0$$

$\lambda_\pm = \pm\sqrt{2(s+\lambda)}$ so $\psi(x) = c_1 e^{-\sqrt{2(s+\lambda)}\,x}$

for $0 < x < \alpha$ $\qquad \frac{1}{2}\psi''(x) - (s)\psi = 0$

So $\psi(x) = c_2 e^{-\sqrt{2s}\,x} + c_3 e^{+\sqrt{2s}\,x}$

for $x < 0$ $\qquad \frac{1}{2}\psi''(x) - s\psi = 0$

$$\psi(x) = c_4 e^{-\sqrt{2s}\,x}$$

3. $\Rightarrow$ $\qquad c_4 = c_2 + c_3$ at $x = 0$

5. $\Rightarrow$ $\qquad -\sqrt{2s}\,c_4 = -\sqrt{2s}\,c_2 + \sqrt{2s}\,c_3 + 2$

$c_4 = c_2 - c_3 - \sqrt{\frac{2}{s}}$ $\qquad$ so $2c_3 + \sqrt{\frac{2}{s}} = 0$

$c_3 = -\frac{1}{\sqrt{2s}}$

3. $\Rightarrow$ $\qquad c_1 e^{-\sqrt{2(s+\lambda)}\,\alpha} = c_2 e^{-\sqrt{2s}\,\alpha} + c_3 e^{+\sqrt{2s}\,\alpha}$

$$4. \Rightarrow \quad c_1\left(-\sqrt{2(s+\lambda)}\right)e^{-\sqrt{2(s+\lambda)}\,\alpha} = -\sqrt{2s}\,c_2 e^{-\sqrt{2s}\,\alpha} + \sqrt{2s}\,c_3 e^{+\sqrt{2s}\,\alpha}$$

$$= \sqrt{2s}\left(c_3 e^{\sqrt{2s}\,\alpha} - c_2 e^{-\sqrt{2s}\,\alpha}\right)$$

So $$\frac{-\sqrt{2(s+\lambda)}}{\sqrt{2s}}\,c_1 e^{-\sqrt{2(s+\lambda)}\,\alpha} = c_3 e^{+\sqrt{2s}\,\alpha} - c_2 e^{-\sqrt{2s}\,\alpha}$$

$$c_1 e^{-\sqrt{2(s+\lambda)}\,\alpha} = c_3 e^{+\sqrt{2s}\,\alpha} + c_2 e^{-\sqrt{2s}\,\alpha}$$

$$2c_3 e^{+\sqrt{2s}\,\alpha} = c_1\left(1 - \sqrt{\frac{s+\lambda}{s}}\right)e^{-\sqrt{2(s+\lambda)}\,\alpha}$$

$$c_3 = c_1 \frac{1}{2}\left(1 - \sqrt{\frac{s+\lambda}{s}}\right)e^{-\left(\sqrt{2(s+\lambda)} + \sqrt{2s}\right)\alpha}$$

So $$c_1 = \frac{2}{1 - \sqrt{\frac{s+\lambda}{s}}}\,e^{\left(\sqrt{2(s+\lambda)} + \sqrt{2s}\right)\alpha}\,c_3 \qquad c_3 = \frac{-1}{\sqrt{2s}}$$

$$= \left(\frac{-\sqrt{2}}{\sqrt{s} - \sqrt{s+\lambda}}\right)e^{\left(\sqrt{2(s+\lambda)} + \sqrt{2s}\right)\alpha}$$

$$e^{-\sqrt{2(s+\lambda)}\,\alpha}\left(\frac{-\sqrt{2}}{\sqrt{s} - \sqrt{s+\lambda}}\right)e^{\sqrt{2(s+\lambda)}\,\alpha}\,e^{\sqrt{2s}\,\alpha} = c_2 e^{-\sqrt{2s}\,\alpha} - \frac{1}{\sqrt{2s}}\,e^{\sqrt{2s}\,\alpha}$$

$$\frac{-\sqrt{2}}{\sqrt{s} - \sqrt{s+\lambda}} = c_2 e^{-2\sqrt{2s}\,\alpha}$$

$$c_2 = \frac{-\sqrt{2}}{\sqrt{s} - \sqrt{s+\lambda}}\,e^{2\sqrt{2s}\,\alpha}$$

So $$c_4 = \frac{-1}{\sqrt{2s}} + \frac{-\sqrt{2}}{\sqrt{s} - \sqrt{s+\lambda}}\,e^{2\sqrt{2s}\,\alpha}$$

$$\cancel{= -1\left(\frac{\frac{1}{\sqrt{2}}}{\sqrt{s}} + \frac{\sqrt{2}}{\sqrt{s} - \sqrt{s+\lambda}}\right)e^{2\sqrt{2s}\,\alpha}}$$

Whoops...

$$-1\left(\frac{1}{\sqrt{2s}} + \frac{\sqrt{2}}{\sqrt{s} + \sqrt{s+\lambda}}\,e^{2\sqrt{2s}\,\alpha}\right).$$

11/23

Prob.

2. Strong Limit Theorems

Weak law actually!

Strong Law of Large Number: Bernoulli Let $X_1, X_2, \cdots$ be i.i.d. Bernoulli r.v.'s $S_n = X_1 + \cdots + X_n$. $P\{X_i = 1\} = p$
$P\{X_i = 0\} = q = 1-p$

$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P\{|\frac{S_n}{n} - P| \geq \varepsilon\} = 0$ convergence in measure.

<u>Theorem</u>:
Let $X_1, X_2, \cdots$ be i.i.d. r.v.'s with mean μ and variance σ^2, $S_n = \sum_{i=1}^n X_i$
then $\forall \varepsilon > 0 \quad \lim_{n\to\infty} P\{|\frac{S_n}{n} - P| \geq \varepsilon\} = 0$. (Note p & μ are the same)

<u>Proof</u>: Reminder of Chebychev inequality
X has mean μ variance then $P\{|X-\mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$

$$\sigma^2 = \int (X(\omega) - \mu)^2 dP \geq \varepsilon^2 \int_{|X(\omega) - \mu| \geq \varepsilon} dP = \varepsilon^2 P\{|X-\mu| \geq \varepsilon\}.$$

So $P\{|\frac{S_n}{n} - P| \geq \varepsilon\} \leq \frac{\sigma^2}{n\varepsilon^2} \to 0$ as $n \to \infty$. Very trivially

Steinhause (1922) first people to state probability.
Borel (1909) $.a_1 a_2 \cdots$ ← binary expansion for $x \in (0,1)$

$$P\{\lim_{n\to\infty} \frac{\sum a_i}{n} = \frac{1}{2}\} = 1.$$

Consider $(\Omega, \mathcal{B}, P)$

<u>Theorem</u>: Kolmogorov's Inequality. Let $X_1, X_2, \cdots, X_n$ be ind. r.v.'s with mean 0 and variances σ_k^2, $1 \leq k \leq n$. Let $S_k = \sum_{j=1}^{k} X_j$.

$$P\{\max(|S_1|, \cdots, |S_n|) \geq \varepsilon\} \leq \frac{\sum_{k=1}^n \sigma_k^2}{\varepsilon^2}$$

Remark: Chebychev's inequality applied to S_n.

$$P\{|S_n| \geq \varepsilon\} \leq \frac{\sum_{k=1}^{n} \sigma_k^2}{\varepsilon^2}$$

<u>Proof</u>: Let $E = \{\max(|S_1|, |S_2|, \cdots, |S_n|) \geq \varepsilon\}$ (a set in Ω)

$E_k = \{\omega \in \Omega : |S_k| \geq \varepsilon,\ |S_i| < \varepsilon \quad 1 \leq i < k\}$

note $E = E_1 \cup E_2 \cdots \cup E_n \qquad E_i \cap E_j = \phi \quad i \neq j$ therefore

$P(E) = \sum_{k=1}^{n} P(E_k)$. Consider $\sum_{k=1}^{n} \sigma_k^2 = \sum_{k=1}^{n} \int_\Omega X_k^2(\omega)\, dP =$

$$= \int_\Omega \sum_{k=1}^{n} X_k^2(\omega)\, dP = \int_\Omega \left(\sum_{k=1}^{n} X_k(\omega)\right)^2 dP$$

$$= \int_\Omega S_n^2\, dP \geq \int_E S_n^2\, dP = \sum_{k=1}^{n} \int_{E_k} S_n^2\, dP$$

$$= \sum_{k=1}^{n} \int_{E_k} [S_k - (S_n - S_k)]^2\, dP = \sum_{k=1}^{n} \left[\int_{E_k} S_k^2\, dP + P(E_k) \sum_{j=k+1}^{n} \sigma_j^2\right]$$

$$\geq \sum_{k=1}^{n} \int_{E_k} S_k^2\, dP \qquad \text{on } E_k \quad |S_k| \geq \varepsilon$$

$$\geq \varepsilon^2 \sum_{k=1}^{n} P(E_k) = \varepsilon^2 P(E) \qquad \text{Q.E.D.}$$

<u>Theorem</u>: Let $X_1, X_2, \cdots$ be sequence of indep. r.v.'s with mean 0 and variances σ_k^2. If $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$ then

$$\sum_{k=1}^{\infty} X_k(\omega) \text{ converges a.e. in } (\Omega, \mathcal{B}, P).$$

<u>roof</u>: Let $S_n(\omega) = \sum_{j=1}^{n} X_j(\omega)$ and let $a_m(\omega) = \sup_{k \geq 1} |S_{m+k}(\omega) - S_m(\omega)|$

$\forall$ [illegible] and

$$a(\omega) = \inf_{m \geq 1} a_m(\omega).$$

The n.a.s. condition that the series converge at an ω is

$a(\omega) = 0$

Let $\varepsilon > 0$ be given. For any integers m, n we have from Kolmogorov's inequality

$$P\left\{\max_{1\le k\le n} |S_{m+k}(\omega) - S_m(\omega)| \ge \varepsilon\right\} \le \frac{\sum_{k=m+1}^{m+n} \sigma_k^2}{\varepsilon^2}$$

$$\therefore \quad P\{a_m(\omega) \ge \varepsilon\} \le \frac{\sum_{k=m+1}^{\infty} \sigma_k^2}{\varepsilon^2} < \infty \text{ by the definition as } n\to\infty$$

$$\therefore \quad P\{a(\omega) \ge \varepsilon\} \le \frac{\sum_{k=m+1}^{\infty} \sigma_k^2}{\varepsilon^2} \quad \text{by definition of inf}$$

So for $\forall \varepsilon > 0$ $P\{a(\omega) \ge \varepsilon\} = 0$ as $m \to \infty$ $\therefore P\{a(\omega) = 0\} = 1$.

Application: Recall

$$\sum_{k=0}^{\infty} \frac{\alpha_k}{\sqrt{\rho_k}} u_k(\tau) \qquad \alpha_k \text{ are indep. Gaussian variab}$$

For each fixed τ this is a series $\sum_{k=0}^{\infty} X_k(\omega)$ of ind. r.v.'s with mean zero and with variance $\frac{u_k^2(\tau)}{\rho_k}$. From theorem just proved this series converges for a.all ω if converges

$$\sum_{k=1}^{\infty} \frac{u_k^2(\tau)}{\rho_k} = \tau = \min(\tau, \tau) \text{ from Mercer's theorem}$$

So this in $n(0,\tau)$ and the covariance is $\min(\tau, s)$ so it is another form of Brownian motion.

The little lemmas:

Lemma 1: If $\{y_n\}$ is a sequence of real numbers $\ni \lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} y_j = y$.

Lemma 2: If $\{y_n\}$ is a sequence of real numbers $\ni \sum_{n=1}^{\infty} \frac{y_n}{n} < \infty$ then

$$\lim_{n\to\infty} \frac{y_1 + y_2 + \cdots + y_n}{n} = 0.$$

Theorem: (1st form of Strong L. of L.N.) Let $X_1, X_2, \cdots$ be a sequence of r.v.'s with mean zero and variances $\sigma_k^2 \ni \sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2}$ converges.

Then $P\{\lim_{n\to\infty} \frac{S_n}{n} = 0\} = 1$.

Remark: The r.v.'s have two moments and if they were identically distributed the hypothes is automatic and we have trivially

$$P\{\lim_{n\to\infty} \frac{S_n}{n} = \mu\} = 1.$$

Proof: Let $Y_k = \frac{X_k}{k}$ then the Y_k are indep. have mean zero and variance σ_k^2/k^2 therefore by theorem on infinite series

$\sum_{k=1}^{\infty} Y_k(\omega)$ converges for almost all ω, i.e. $\sum_{k=1}^{\infty} \frac{X_k}{k}$ converg- a.e. then by Lemma 2 $\frac{S_n}{n} \to 0$ a.e. Q.E.D.

Application: Let $X_1, X_2, \cdots$ be i.r.v's having common d.f. $F(x)$, and $r_n(x) = \#$ of X_j's out of $X_1, X_2, \cdots, X_n \leq x$. Then $\forall x$

$P\{\lim_{n\to\infty} \frac{r_n(x)}{n} = F(x)\} = 1$. This is the empirical distribution function, so this keeps statisticians in business.

Proof: Let $Y_j = \begin{cases} 1 & \text{if } X_j \leq x \\ 0 & \text{if } X_j > x \end{cases}$ for a given x fixed. Note that

$Y_1, Y_2, \cdots$ are indep. r.v.'s with mean $F(x)$ and (~~variance~~ second moment) $F(x)$. So by the remark $S_n = \sum_{j=1}^{k} Y_j = r_n(x)$ so $P\{\lim_{n\to\infty} \frac{r_n}{n} = F(x)\} = 1$.

We will prove: Theorem: Let $X_1, X_2, \cdots$ be i.i.d. r.v. with mean zero and variance one. Let N_n = # of partial sums S_j out of $S_1, S_2, \dots, S_n$ which are ≥ 0. Then

$$\lim_{n\to\infty} P\left\{\frac{N_n}{n} < \alpha\right\} = \Sigma(\alpha) = \begin{cases} 0 & \alpha < 0 \\ \frac{2}{\pi}\arcsin\sqrt{\alpha} & 0 \leq \alpha \leq 1 \\ 1 & \alpha \geq 1 \end{cases}$$

(the arcsin law).

Proof: (Using the Feynman-Kac formula): Let $X^{(n)}(\tau) = \begin{cases} \frac{S_1}{\sqrt{n}} & \tau = 0 \\ \frac{S_i}{\sqrt{n}} & \frac{i-1}{n} < \tau \leq \frac{i}{n} \end{cases}$, $0 \leq \tau \leq 1$

this is a random step function. The invariance priciple states that for a large class of functionals $\mathcal{F}$, $F \in \mathcal{F}$

$$\lim_{n\to\infty} P\left\{F[X^{(n)}(\cdot)] \leq \alpha\right\} = P\left\{F[\beta(\cdot)] \leq \alpha\right\}$$

B.M.

For example, let $F[\beta] = \int_0^1 \left(\frac{1+\operatorname{sgn}\beta(s)}{2}\right) ds$ $\quad \operatorname{sgn} x = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$

Then " says $\lim_{n\to\infty} P\left\{\frac{N_n}{n} \leq \alpha\right\} = P\left\{\left(\int_0^1 \frac{1+\operatorname{sgn}\beta(s)}{2} ds\right) \leq \alpha\right\}$

B.M. Lebesgue measure of the B.M. path that is positive. It will suffice to show

$$P\left\{\left(\int_0^1 \frac{1+\operatorname{sgn}\beta(s)}{2} ds\right) \leq \alpha\right\} = \Sigma(\alpha)$$

Let $\sigma(\alpha, t) = P\left\{\int_0^t \frac{1+\operatorname{sgn}\beta(s)}{2} ds \leq \alpha\right\}$ consider for $\lambda > 0$

$$E\left\{e^{-\lambda\int_0^t \left(\frac{1+\operatorname{sgn}\beta(s)}{2}\right) ds}\right\} = \int_0^\infty e^{-\lambda\alpha}\, d\sigma(\alpha, t)$$

Let $u(x, t:\lambda) = E\left\{e^{-\lambda\int_0^t \left(\frac{1+\operatorname{sgn}\beta(s)}{2}\right) ds}\, \delta(\beta(t) - x)\right\}$ therefore

(3) $\int_{-\infty}^{\infty} u(x,t:\lambda)\,dx = \int_0^{\infty} e^{-\lambda\alpha}\,d\sigma(\alpha,t)$. So now we try to find u, from the Feynman-Kac formula

$u(x,t:\lambda)$ satisfies $\quad u_t = \frac{1}{2}u_{xx} - \lambda V(x)u$, $\quad V(x)=\begin{cases}1 & x\geq 0\\ 0 & x<0\end{cases}$

$u(x,0)=\delta(x)$

This is equivalent to the following integral equation

$$u(x,t:\lambda) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} - \lambda\int_0^t d\tau \underbrace{\int_{-\infty}^{\infty} d\xi\, V(\xi)\,u(\xi,\tau:\lambda)\frac{e^{\frac{-(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}}}_{\text{a convolution in } \tau.}$$

Apply $\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$ on both sides, heuristically

$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0 - \lambda V(x)\,u(x,t:\lambda).$$ Let

$$\Psi(x,s:\lambda) = \int_0^{\infty} e^{-st}u(x,t:\lambda)\,dt \qquad s>0.$$ And so to the integral equation:

$$\Psi(x,s:\lambda) = \frac{1}{\sqrt{2s}}e^{-\sqrt{2s}|x|} - \lambda\int_{-\infty}^{\infty} V(\xi)\,\Psi(\xi,s:\lambda)\frac{1}{\sqrt{2s}}e^{-\sqrt{2s}|x-\xi|}\,d\xi$$

But this is equivalent to an o.d.e.

$$\frac{1}{2}\Psi'' - (s+\lambda V(x))\Psi = 0 \qquad \Psi\to 0 \text{ as } x\to\pm\infty$$

Ψ' is continuous at $x\neq 0$

$\Psi'(0^-) - \Psi'(0^+) = 2$. If you solve this and Ψ is continuous at $x=$

$$\Psi(s,x:\lambda) = \begin{cases}\dfrac{\sqrt{2}}{\sqrt{s+\lambda}+\sqrt{s}}\,e^{-\sqrt{2(s+\lambda)}\,x} & x\geq 0\\[2ex] \dfrac{\sqrt{2}}{\sqrt{s+\lambda}+\sqrt{s}}\,e^{\sqrt{2s}\,x} & x<0\end{cases}$$

Therefore $\int_{-\infty}^{\infty} \Psi(s,x:\lambda)\,dx = \frac{1}{\sqrt{s(s+\lambda)}}$ from (2)

$$\int_0^\infty e^{-st}\int_{-\infty}^{\infty} u(x,t:\lambda)\,dx\,dt = \int_0^\infty e^{-st}\int_0^\infty e^{-\lambda\alpha}\,d\sigma(\alpha,t)\,dt$$

by Fubini

$$\frac{1}{\sqrt{s(s+\lambda)}} = \int_{-\infty}^{\infty}\Psi(x,s:\lambda)\,dx = \int_0^\infty e^{-st}\underbrace{\int_0^\infty e^{-\lambda\alpha}\,d\sigma(\alpha,t)}_{F(t)}\,dt$$ Inverse Laplace T.

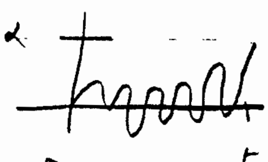
So $F(t) = e^{-\frac{\lambda t}{2}} \underbrace{I_0\left(\frac{\lambda t}{2}\right)}_{\text{a Bessel Function}} = \int_0^\infty \sigma'(\alpha,t)\,d\alpha\, e^{-\lambda\alpha}$ so Inverse Laplace T

$$\sigma'(\alpha,t) = \begin{cases} \frac{1}{\pi\sqrt{\alpha(t-\alpha)}} & 0<\alpha<t \\ 0 & \alpha>t \end{cases}$$

So $\sigma(\alpha,t) = \begin{cases} 0 & \alpha\le 0 \\ \frac{2}{\pi}\arcsin\sqrt{\frac{\alpha}{t}} & 0\le\alpha\le t \\ 1 & \alpha\ge t\end{cases}$ set $t=1$. Q.E.D.

This theorem was first proved this way by Kac etc. This is a combinatorial result, actually.

<u>Problem</u>: $P\{\max\limits_{0\le s\le t}\beta(s)\le\alpha\}$



$= \lim\limits_{n\to\infty} P\left\{\max\left(\frac{S_1}{\sqrt{n}},\frac{S_2}{\sqrt{n}},\dots,\frac{S_n}{\sqrt{n}}\right)\le\alpha\right\} = H(\alpha)$. Let

(β is continuous so if $\max\limits_{0\le s\le t}\beta(s)\le\alpha$ then $V_\alpha(\beta(s))=1$ on a set of positive measure) so $V_\alpha(x) = \begin{cases} 1 & x\geq\alpha \\ 0 & x<\alpha\end{cases}$ let $\lambda>0$ and consider

$$\lim_{\lambda\to\infty} E\left\{e^{-\lambda\int_0^t V_\alpha(\beta(s))\,ds}\right\} = H(\alpha)$$ because for $\beta(s)\ge\alpha$ as $\lambda\to 0$ $e^{-\lambda\int_0^t V}=0$ for $\beta(s)<\alpha$ as $\lambda\to 0$ $e^{-\lambda\int_0^t}$

this is calculable by Feynman-Kac

$$H(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\frac{\alpha}{\sqrt{t}}} e^{-u^2/2}\,du.$$

1/30 ①

Prob.

Thrm. 1: If $X_1, X_2, \cdots$ are i.r.v.'s with mean 0 and variances $\sigma_k^2 \ni \sum_{k=1}^{\infty} \sigma_k^2 < \infty$ then $\sum_{n=1}^{\infty} X_n$ converges a.e.

Thrm. 2: If $X_1, X_2, \cdots$ be i.r.v.'s for which $\exists\, c < \infty$ a constant $\ni |X_n| \le c$ a.e. in Ω $n = 1, 2, \cdots$ then if $\sum_{n=1}^{\infty} X_n$ converges on a set of positive measure then $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$. (Also $E(X_i) = 0$)

Corr: If $\{X_n\}$ is an essentially uniformly bounded sequence of i.r.v.'s then $\sum_{i=1}^{\infty} X_i$ either converges or diverges a.e., the X_i's have mean zero.

Compare later with the n zero-one law.

Proof: Let $S_0 = 0$ $S_n = \sum_{k=1} X_k$. Let $G \subset \Omega$ where the series converges, i.e. on G $\lim_{n \to \infty} S_n$ exists and $P(G) > 0$.

By Egorov's theorem $\exists\, K > 0 \ni E = \{\omega : |S_n(\omega)| \le K \ \forall n\}$
$= \bigcap_{n=0}^{\infty} \{\omega : |S_n(\omega)| \le K\}$ has positive measure, i.e. $P(E) > 0$. Let $E_n = \bigcap_{k=0}^{n} \{\omega : |S_k(\omega)| \le K\}$

Then $\{E_n\}$ is a decreasing sequence of sets $\ni \bigcap_{n=0}^{\infty} E_n = E$. Consider $n = 1, 2, \cdots$

$$\int_{E_n} S_n^2 \, dP - \int_{E_{n-1}} S_{n-1}^2 \, dP = \int_{E_{n-1}} S_n^2 \, dP - \int_{E_{n-1}} S_{n-1}^2 \, dP$$

$$- \int_{E_{n-1} - E_n} S_n^2 \, dP$$

$$= \int (X_n + S_{n-1})^2 \, dP - \int S_{n-1}^2 \, dP - \int S_n^2 \, dP$$

split it up and reap great rewards

$$= \int_{E_{n-1}} X_n^2 dP + 2\int_{E_{n-1}} X_n S_{n-1} dP - \int_{E_{n-1}-E_n} S_n^2 dP$$

$$= \sigma_n^2 P(E_{n-1}) + 0 - \int_{E_{n-1}-E_n} S_n^2 dP = \sigma_n^2 P(E_{n-1}) - \int_{E_{n-1}-E_n} (X_n + S_{n-1})^2 dP$$

on E_{n-1}; $|S_{n-1}(\omega)| \le K$ and $|X_n| \le c$ a.e.

$\therefore$ $|X_n + S_{n-1}| \le c + K$ a.e. on E_{n-1}

$$\ge \sigma_n^2 P(E_{n-1}) - (c+K)^2 P(E_{n-1} - E_n)$$
$$\ge \sigma_n^2 P(E) - (c+K)^2 P(E_{n-1} - E_n)$$

So summing from $n=1$ to N because of the telescopic series

$$\int_{E_N} S_N^2 dP \ge P(E)\sum_{i=1}^{N} \sigma_i^2 - (c+K)^2. \text{ But also}$$

$$\int_{E_N} S_N^2 dP \le K^2 P(E_n) \le K^2. \text{ So}$$

$$\sum_{i=1}^{N} \sigma_i^2 \le \frac{K^2 + (c+K)^2}{P(E)}$$ so the series converges. Q.E.D.

Theorem 3: Let $\{X_n\}$ be i.r.v.'s for which $\exists c > 0 \ni$ $|X_n| \le c$ a.e. $n = 1, 2, \ldots$ then $\sum_{i=1}^{\infty} X_n$ converges a.e. iff both series

$$\sum_{n=1}^{\infty} \mu_n \quad \& \quad \sum_{n=1}^{\infty} \sigma_n^2$$ converge where μ_n, σ_n^2 are the mean and variances of X_n.

Proof: If the series converge and $|X_n| \le c$ $\forall n = 1, 2, 3, \ldots$ then $Y_n = X_n - \mu_n$ are independent r.v.'s and

$\sum_{n=1}^{\infty} \sigma_n^2(Y_n) = \sum_{n=1}^{\infty} \sigma_n^2 < \infty$ we have by theorem 1 that $\sum_{n=1}^{\infty} Y_n$ converges a.e. but $\sum_{n=1}^{\infty} \mu_n$ converges so $\sum_{n=1}^{\infty} X_n$ converges a.e.

<u>Other way</u>: Consider $\Omega \times \Omega$ with measure $P \times P$ let

$$h_n(\omega_1, \omega_2) = X_n(\omega_1) - X_n(\omega_2) \qquad n = 1, 2, 3, \dots$$

From our hypothesis $\sum_{n=1}^{\infty} h_n(\omega_1, \omega_2)$ converges a.e. on $\Omega \times \Omega$, moreover

$$\int_{\Omega \times \Omega} h_n(\omega_1, \omega_2)\, dP \times dP = \int_{\Omega} X_n\, dP - \int_{\Omega} X_n\, dP = 0$$

also $|h_n| \leq 2c$ a.e. on $\Omega \times \Omega$. By theorem 2 the series $\sum_{n=1}^{\infty} \sigma^2(h_n)$ converges.

$$\int_{\Omega \times \Omega} h_n^2\, d(P \times P) = \int_{\Omega \times \Omega} [X_n^2(\omega_1) - 2X_n(\omega_1)X_n(\omega_2) + X_n^2(\omega_2)]\, d(P \times P)$$

$$= \sigma_n^2 + \mu_n^2 - 2\mu_n^2 + \sigma_n^2 + \mu_n^2 = 2\sigma_n^2$$

therefor $\sum_{n=1}^{\infty} \sigma^2(h_n) = 2\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Let $Y_n = X_n - \mu_n$ then

$\sigma^2(Y_n) = \sigma_n^2 \quad \therefore \sum_{n=1}^{\infty} \sigma^2(Y_n) < \infty$ this implies that

$\sum_{n=1}^{\infty} Y_n$ converges a.e. but then $\sum_{n=1}^{\infty} \mu_n < \infty$.

[illegible] Y_n converge a.e. and $Y_n = X_n - \mu_n$.

[illegible]). Let X, Y [illegible] be a [illegible]. Let $c > 0$ be any constant. A necessary and sufficient condition that $\sum_{n=1}^{\infty} X_n$ converges a.e. is that the three series convergerge:

1) $\sum_{n=1}^{\infty} P\{\omega : |X_n(\omega)| > c\} = \sum_{n=1}^{\infty} \int_{|x|>c} dF_n(x)$

2) $\sum_{n=1}^{\infty} \int_{|X_n(\omega)| \le c} X_n(\omega)\, dP = \sum_{n=1}^{\infty} \int_{|x| \le c} x\, dF_n(x)$ truncated mean

3) $\sum_{n=1}^{\infty} \left(\int_{|X_n(\omega)| \le c} X_n^2(\omega)\, dP - \left[\int_{|X_n(\omega)| \le c} X_n(\omega)\, dP\right]^2 \right)$ truncated variance.

$$= \sum_{n=1}^{\infty} \left[\int_{|x| \le c} x^2\, dF_n(x) - \left(\int_{|x| \le c} X_n(\omega)\, dP \right)^2 \right]$$

Proof: Let $Y_n = \begin{cases} X_n & \text{if } |X_n| \le c \\ c & \text{if } |X_n| > c \end{cases}$

$Z_n = \begin{cases} X_n & \text{if } |X_n| \le c \\ -c & \text{if } |X_n| > c \end{cases}$ It is obvious that $\sum_{n=1}^{\infty} X_n, \sum_{n=1}^{\infty} Y_n, \sum_{n=1}^{\infty} Z_n$ converge at exactly the same points.

From theorem 3 a n.s. for $\sum_{n=1}^{\infty} Y_n$ converges a.e. is that $\sum_{n=1}^{\infty} E(Y_n) < \infty$, $\sum_{n=1}^{\infty} \sigma^2(Y_n) < \infty$, i.e.

$$E(Y_n) = \int_{|X_n| \le c} X_n(\omega)\, dP \pm c\, P\{|X_n(\omega)| > c\}$$

$$\sum_{n=1}^{\infty} \left(\int_{|X_n| \le c} X_n(\omega)\, dP \pm c\, P\{|X_n(\omega)| > c\} \right) < \infty \quad \text{and}$$

$$\sum_{n=1}^{\infty} \left(\int_{|X_n| \le c} X_n^2(\omega)\, dP - \left(\int_{|X_n| \le c} X_n(\omega)\, dP\right)^2 + c^2 P\{|X_n| \le c\} P\{|X_n| > c \right.$$

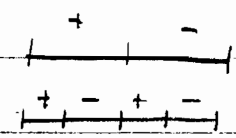
$$\mp 2c\, P\{|X_n| > c\} \int_{|X_n| \le c} X_n\, dP ,$$

So we have four series consider the $\sum_{n=1}^{\infty} Z_n$ by theorem 3. adding and subtracting we get this because theorem 3 is nts.

If you have i.r.v.'s $\sum_{n=1}^{\infty} X_n$ converges or diverges a.e. Some fun with problems. Let $\Omega = [0,1]$ $\mathcal{B}$ be Lebesgue measurable sets on $[0,1]$ and P is Lebesgue measure. So $\omega \in [0,1]$.

$$\text{Let} \quad X_n(\omega) = \begin{cases} 1 & \text{if } i \text{ is odd } \left(\frac{i-1}{2} \le \omega < \frac{i}{2^n}\right) \\ -1 & \text{if } i \text{ is even} \end{cases}$$

$$X_1(\omega) = \begin{cases} 1 & \text{if} \quad 0 \le \omega < \frac{1}{2} \\ -1 & \text{if} \quad \frac{1}{2} \le \omega \le 1 \end{cases}$$



X_1

These are Rademacher functions. Show that $X_1, X_2, X_3, \dots$ is a sequence of independent r.v.'s. Consider $\{a_n\}$ a bounded sequence of real numbers and consider

$$\sum_{n=1}^{\infty} a_n X_n(\omega) \qquad \text{if } Y_n = a_n X_n(\omega)$$

$$E(Y_n) = 0$$

$\sigma^2(Y_n) = a_n^2$ $\therefore$ from Theorem 3 a n.s. condition for the convergence $\sum_{n=1}^{\infty} a_n X_n(\omega)$ a.e. is $\sum_{n=1}^{\infty} a_n^2$ converge.

So when does $\sum_{n=1}^{\infty} \pm a_n$ converge Ans When $\sum_{n=1}^{\infty} a_n^2 < \infty$.

random coin flips determine this, but this is done by the Rademacher functions.

Brownian motion Heuristics: (A new formalism to play with)

With $\beta(s)$. B.M $0 \le s \le t$ $\beta(0)=0$.

$E\{F[\beta]\}$ if $f(u_1, u_2, \dots, u_n)$ is a Borel measurable function on $\mathbb{R}^n$ and if $0 < \tau_1 < \tau_2 < \dots < \tau_n \le t$ then

$$E\{f(\beta(\tau_1), \dots, \beta(\tau_n))\} = \frac{1}{\sqrt{(2\pi)^n \tau_1 \cdots (\tau_n - \tau_{n-1})}} \int_{-\infty}^{\infty}\!\!\overset{(n)}{\cdots}\!\!\int f(u_1, \dots, u_n)\, e^{-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{\tau_j - \tau_{j-1}}} du_1 \cdots du_n$$

$u_0 = \tau_0 = 0$

For every path $\beta(\cdot)$ on $[0,t]$ evaluate at $\frac{jt}{n}$ $j = 0, 1, 2, \dots, n$ and linearize in between. For nice functionals we expect

$$E\{F[\beta]\} = \lim_{n\to\infty} \frac{1}{\sqrt{(2\pi)^n \tau_1 \cdots (\tau_n - \tau_{n-1})}} \int_{-\infty}^{\infty}\!\!\overset{(n)}{\cdots}\!\!\int F_n(u_1, \dots, u_n)\, e^{-\frac{1}{2}\sum \frac{(u_j - u_{j-1})^2}{(\tau_j - \tau_{j-1})}} du_1 \cdots du_n$$

Von Neumann has proved that there is no Haar measure in function space, translational invariance that is. So suppose

$$E\{F[\beta(\cdot)]\} \text{ "=" } \overset{\text{flat integral}}{\int} F[\beta]\, e^{\underbrace{-\frac{1}{2}\int_0^t [\beta'(\tau)]^2 d\tau}_{\text{this is the action } A[\beta]}} \, \underset{\text{element in function space.}}{\delta\beta}$$

Consider for $\varepsilon > 0$ $E\left\{e^{\frac{1}{\sqrt{\varepsilon}}\int_0^t \beta(s)ds}\right\}$

$$= \prod_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{\alpha}{\sqrt{\varepsilon}}\int_0^t u_k(s)ds - \frac{\alpha^2}{2}} d\alpha$$

$$= e^{\frac{1}{6\varepsilon}}$$

$\therefore$ $\lim_{\varepsilon\to 0} \varepsilon \log E\left\{e^{\frac{1}{\sqrt{\varepsilon}}\int_0^t \beta(s)ds}\right\} = \frac{1}{6}$. Consider

$E\left\{e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta]}\right\}$, how does this behave is as $\varepsilon \to 0$?

$$\text{“}=\text{”} \int e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta]} e^{-\frac{1}{2}\int_0^t [\beta'(s)]^2 ds} \delta\beta \qquad \text{Let } \sqrt{\varepsilon}\beta = w$$

$$\text{“}=\text{”} \int e^{\frac{1}{\varepsilon}\left[F[w] - \frac{1}{2}\int_0^t [w'(s)]^2 ds\right]} \delta\beta \qquad \text{by Laplace}$$

$$\sim e^{\frac{1}{\varepsilon}\sup_{w \in C_0^*[0,t]} \left[F(w) - \frac{1}{2}\int_0^t [w'(s)]^2 ds\right]}$$

$C_0^*[0,t]$ ← vanishing at 0, continuous with derivatives in L^2.

Conjecture

$$\lim_{\varepsilon \to 0} \varepsilon \log E\left\{e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta]}\right\} = \sup_{w \in C_0^*[0,t]} \left(F[w] - \frac{1}{2}\int_0^t [w'(s)]^2 ds\right) \text{ and}$$

three cleverly chosen examples should be done.

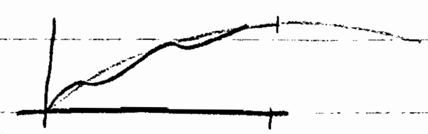
1). $F[\beta] = \int_0^t \beta(s)\, ds$

$$E\left\{e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta]}\right\} = E\left\{e^{\frac{1}{\sqrt{\varepsilon}}\int_0^t \beta(s) ds}\right\} \text{ from conjecture}$$

essentially arc length

$$\lim_{\varepsilon \to 0} \varepsilon \log E\left\{e^{\frac{1}{\sqrt{\varepsilon}}\int_0^t \beta(s) ds}\right\} = \sup_{w \in C_0^*(0,t)} \left[\int_0^t w(s) ds - \frac{1}{2}\int_0^t [w'(s)]^2 ds\right]$$

$$= \frac{1}{6}.$$



By Calculus of variations the Euler equation is

$$1 + w''(s) = 0$$
$$w(0) = 0 \qquad \Rightarrow \quad w(s) = \frac{-s^2}{2} + ts$$
$$w'(t) = 0$$

So it works, woopee-doopee. This is action asymptotics.

$E\{F[\beta]\}$ $\quad y(\cdot) \in C_0(0,t)$ and we want

$$E\{F[\beta+y]\} \text{“}=\text{”} \int F[\beta+y]\, e^{-\frac{1}{2}\int_0^t [\beta'(s)]^2 ds}\, \delta\beta \qquad \text{let } \beta+y = w$$

$$\text{“}=\text{”} \int F[w]\, e^{-\frac{1}{2}\int_0^t [w'(s)-y'(s)]^2 ds}\, \delta w$$

$$\text{“}=\text{”} e^{-\frac{1}{2}\int_0^t [y'(s)]^2 ds} \int F[w]\, e^{+\int_0^t y'(s) w'(s) ds - \frac{1}{2}\int_0^t [w'(s)]^2 ds}\, \delta w$$

B.V. cont.

$$\text{“}=\text{”} e^{-\frac{1}{2}\int_0^t [y'(s)]^2 ds}\, E\left\{F[\beta]\, e^{\int_0^t y'(s)\, d\beta(s)}\right\}$$

∘ conjecture $\quad E\{F[\beta+y]\} = e^{-\frac{1}{2}\int_0^t [y'(s)]^2 ds}\, E\left\{F[\beta]\, e^{\int_0^t y'(s)\, d\beta(s)}\right\}$

where $y \in$ B.V.

12/7

Prob.

①

Lemma 1: The first Borel-Cantelli lemma; With $(\Omega, \mathcal{B}, P)$,

If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P\{\limsup_{n\to\infty} A_n\} = 0$ with $A_n \in \mathcal{B}$ $n=1,2,\cdots$

⊛ $\limsup_{n\to\infty} A_n$ = set of points that belong to infinitely many
$\liminf_{n\to\infty} A_n$ = " " " " " " all but a finite number

so $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$ ⊛ the event A_n occur infinitly often.

Proof: Let $A = \limsup_{n\to\infty} A_n$ then $A \subset \bigcup_{n=m}^{\infty} A_n$ $\forall m=1,2,3,\cdots$

Therefore $P(A) \le \sum_{n=m}^{\infty} P(A_n)$ $\forall m=1,2,3,\dots$ Q.E.D

Lemma 2: The second Borel-Cantelli lemma; If the series

$\sum_{n=1}^{\infty} P(A_n)$ diverges and if the $\{A_n\}$ are

independent then $P\{\limsup_{n\to\infty} A_n\} = 1$

Proof: Let $A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m$ then

$$1 - P(A) = P(A^c) = P\left(\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m^c\right)$$

But $\bigcap_{m=n}^{\infty} A_m^c = \left(\bigcup_{m=n}^{\infty} A_m\right)^c$ is an increasing sequence of sets in n.

So $1 - P(A) = \lim_{n\to\infty} P\left(\bigcap_{m=n}^{\infty} A_m^c\right)$ (decreasing

$$= \lim \lim P\left(\bigcap^{N} A_m^c\right)$$

by independence $= \lim_{n\to\infty} \lim_{N\to\infty} \prod_{m=n}^{N} (1-P(A_m))$. It is an elementary exercise in advanced calculus that if $0 \le a_n \le 1$ and $\sum_{n=1}^{\infty} a_n = \infty$ then $\lim_{n\to\infty} \prod_{j=1}^{n} (1-a_j) = 0$. So for each n $\lim_{N\to\infty} \prod_{m=n}^{N} (1-P(A_m)) = 0 \Rightarrow 1-P(A) = 0$ Q.E.D. Thus we have two Borel-Cantelli lemmas.

We have proved (1^{st} form of SLLN): Let X_n be a sequence of r.v.'s with mean zero and finite variances σ_n^2. If

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n} < \infty \text{ the } P\left\{\lim_{n\to\infty} \frac{S_n}{n} = 0\right\} = 1.$$

We will prove a much nicer theorem.

Theorem: (Kolmogorov law of large numbers): Let $X_1, X_2, \cdots$ be i.i.d.r.v. which are integrable (has a first moment). Let μ be the mean of them. Then

$$P\left\{\lim_{n\to\infty} \frac{S_n}{n} = \mu\right\} = 1.$$

(They are identically distributed). This is a sharp theorem.

Proof: Let $F(x)$ be the common d.f. Define $Y_n = \begin{cases} X_n & \text{if } |X_n| < n \\ 0 & \text{otherwise} \end{cases}$

so $\{Y_n\}$ is a sequence of random variables. Now the variance of Y_n can be calculated

$$\sigma^2(Y_n) \le E(Y_n^2) = \int_{\omega: |X_n(\omega)| < n} X_n^2(\omega)\, dP = \int_{|x|<n} x^2\, dF(x)$$

$= \sum^{n} \int \ldots dF \ldots$

12/7

Prob.

$$\sum_{n=1}^{\infty} \frac{\sigma^2(Y_n)}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{j=1}^{\infty} \int_{j-1 \le |x| < j} x^2 \, dF(x) \right) = \sum_{j=1}^{\infty} \left(\sum_{n=j}^{\infty} \frac{1}{n^2} \right) \left(\int_{j-1 \le |x| < j} x^2 \, dF(x) \right)$$

But $\sum_{n=j}^{\infty} \frac{1}{n^2} < \frac{1}{j^2} + \frac{1}{j(j+1)} + \frac{1}{(j+1)(j+2)} + \cdots = \frac{1}{j^2} + \frac{1}{j} \le \frac{2}{j}$ so

$$\sum_{n=1}^{\infty} \frac{\sigma^2(Y_n)}{n^2} \le 2 \sum_{j=1}^{\infty} \frac{1}{j} \int_{j-1 \le x < j} x^2 \, dF(x) \le 2 \sum_{j=1}^{\infty} \int_{j-1 \le x < j} |x| \, dF(x) = 2 \int_{-\infty}^{\infty} |x| \, dF(x) < \infty$$

Therefore by the first form of the strong law of large numbers

$P\left\{ \lim_{n\to\infty} \sum_{j=1}^{n} \frac{Y_j(\omega) - E(Y_j(\omega))}{n} = 0 \right\} = 1$. Let A be the set

Ω such that $\forall \omega \in A$ $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{Y_j(\omega) - E(Y_j(\omega))}{n} = 0$. So $P(A) = 1$. (2)

et $C_n = \{ \omega : X_n(\omega) \ne Y_n(\omega) \} = \{ \omega : |X_n(\omega)| \ge n \}$ let

$$C = \limsup_{n\to\infty} C_n = \{ \omega : |X_n(\omega)| \ge n \text{ for infinitely many } n \text{'s} \}$$

om the first Borel-Cantelli lemma if we can show $\sum_{n=1}^{\infty} P(C_n) < \infty$ hen we will know $P(C) = 0$.

$$P(C_n) = \int_{|x| \ge n} dF(x) = \sum_{j=n+1}^{\infty} \int_{j-1 \le |x| < j} dF(x) \quad \text{therefore}$$

$$\sum_{n=1}^{\infty} P(C_n) = \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} \int_{j-1 \le |x| < j} dF(x) = \sum_{j=2}^{\infty} (j-1) \int_{j-1 \le |x| < j} dF(x)$$

$$\le \sum_{j=2}^{\infty} \int_{j-1 \le |x| < j} |x| \, dF(x) \le \int_{-\infty}^{\infty} |x| \, dF(x) < \infty$$ by hypothesis so

the series converges and so $P(C)=0$. Let $B=C^c$. Then $P(B)=1$. $B=\{w: X_n \neq Y_n$ for at most a finite number of indices$\}$

Also $P(A\cap B)=1$ now for any $w\in A\cap B$ from (2) and the definition of B

$$(3)\quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n} [X_i(w)-E[Y_i(w)]]}{n} = 0. \quad \text{Because } P(C)=0.$$

We show now $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{E(Y_i)-E(X_i)}{n} = 0$ (4) and that will finish the proof.

Now $E(Y_n) = \int_{|x|<x} x\,dF(x) = \int_{-\infty}^{\infty} x\, I_{(-n,n)}(x)\, dF(x)$ therefore by the Lebesgue dominated convergence theorem

$$\lim E(Y_n) = \int_{-\infty}^{\infty} \lim_{n\to\infty} x\, I_{(-n,n)}\, dF(x) = \mu$$

$\therefore \lim_{n\to\infty} (E(Y_n)-\mu) = 0$ so by a little lemma:

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} [E(Y_j)-\mu]}{n} = 0$$

Excercise: Apply this theorem to the Rademacher functions and you get the famous theorem of Borel (1909).

Consider the interval $[0,1]$ and for each $x\in[0,1]$ in its binary expansion. $x=.a_1a_2a_3\cdots$ where $a_j = \begin{cases} +1 \\ -1 \end{cases}$ if S_n is the excess in the first n binary places of plus ones over minus ones we have $\lim_{n\to\infty} \frac{S_n}{n} = 0$ a.e. in $[0,1]$. Borel in 1909 "almost

Hausdorf (1913) proved $S_n = o(n^{\frac{1}{2}+\epsilon})$ $\forall \epsilon > 0$.
Hardy & Littlewood (1914) proved $S_n = O(\sqrt{n \log n})$
Steinhaus (1922) $\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n\log n}} \le 1$

Khintchine (1923) $S_n = O(\sqrt{n \log\log n})$
Khintchine (1924)
$$\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n\log\log n}} = 1.$$
The law of the iterated logarithm.
How precise is this statement?

If $\lambda > 1$ then for almost all $x \in [0,1]$ $\exists N \ni n > N \Rightarrow S_n \le \lambda\sqrt{2n\log\log n}$
and if $\lambda < 1$ the for almost all $x \in [0,1]$ $\exists$ infinitely many $n \ni$
$S_n \ge \lambda\sqrt{2n\log\log n}$.

Definition: We say $\phi(t)$ belongs to the upper class if $S_n > \sqrt{n}\,\phi(n)$ for only finitely many n. We say $\phi(t)$ belongs to the lower class if $S_n > \sqrt{n}\,\phi(n)$ for infinitely many n. In this language Khintchine's law of the iterated logarithm
$$\lambda\sqrt{2\log\log t} \in \begin{cases} \text{upper class if } \lambda > 1 \\ \text{lower '' '' } \lambda < 1 \end{cases}$$
Paul Lévy (1933)
$$\sqrt{2\log\log t + a\log\log\log t} \in \begin{cases} \text{upper class if } a > 3 \\ \text{lower '' } a \le 1 \end{cases}$$

Kolmogorov (stated) & Erdös (proved) (1937-1942). If $\phi(t)$ is non-decreasing
$$\phi(t) \in \begin{cases} \text{upper class} \\ \text{lower class} \end{cases} \iff \int^{\infty} \frac{1}{t}\phi(t) e^{-\frac{1}{2}\phi^2(t)}\,dt \begin{cases} \text{converge} \\ \text{diverges} \end{cases}$$
? choose a lower limit so $\frac{1}{t}\phi(t)$ isn't too bad.

Hartman-Wintner 1941
Feller 1943
Chung 1948
Strassen 1964 We will get the Khintchine theorem and another extension + then Strassen.

Brownian Motion: $E\{F[\beta]\} \text{``}=\text{''} \int F[\beta] e^{-\frac{1}{2}\int_0^t [\beta'(s)]^2 ds} \delta\beta$ using this formalism we guessed at

$$\lim_{\varepsilon \to 0} \varepsilon \log E\{e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta]}\} = \sup_{w \in C_0^*(0,t)} \left[F(w) - \frac{1}{2}\int_0^t [w'(s)]^2 ds\right]$$

Example: $F[\beta] = \int_0^t \beta(s) ds$ by explicit calculation

$$E\{e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta]}\} = E\{e^{\frac{1}{\sqrt{\varepsilon}}\int_0^t \beta(s) ds}\} = e^{\frac{t^3}{6\varepsilon}}$$

$\therefore \lim_{\varepsilon \to 0} \varepsilon \log E\{e^{\frac{1}{\sqrt{\varepsilon}}\int_0^t \beta(s) ds}\} = \frac{t^3}{6}$. The check is to calculate

$$\sup_{w \in C_0^*(0,t)} \left[\int_0^t w(s) ds - \frac{1}{2}\int_0^t [w'(s)]^2 ds\right]$$

$C_0^*(0,t)$ continuous with derivative in $\mathcal{L}^2$ and vanishing at the origin.

Frechet differential: Suppose $F: C_0(0,t) \to \mathbb{R}$. Let $\psi \in C_0(0,t)$

$$\delta F = \frac{d}{dh} F[\beta + h\psi]\Big|_{h=0}$$

Suppose $F[\beta] = \int_0^t \beta(s)^2 ds$ $\qquad \delta F = 2\int_0^t \beta(\tau)\psi(\tau) d\tau$

$= \int_0^t \frac{\delta F}{\delta \beta(\tau)} \psi(\tau) d\tau$ the old function of n-

In the finite case

$\delta F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \delta x_i$. So here $\frac{\delta F}{\delta\beta(\tau)} = 2\beta(\tau)$. Lets apply such method.

So differentiating $\frac{\delta \int_0^t v(s)\,ds}{\delta v(\tau)} = 1$ and

$$G[v] = -\frac{1}{2}\int_0^t [v'(s)]^2\,ds \,, \qquad G[v+h\psi] = -\frac{1}{2}\int_0^t [v'(s)+h\psi'(s)]^2\,ds$$

$$\delta G = \frac{d\,G[v+h\psi]}{dh}\Big|_{h=0} = -\int_0^t v'(s)\,\psi'(s)\,ds \quad \text{integrate by parts}$$

$$= -\int_0^t v'(s)\,d\psi(s) = \int_0^t v''(s)\,\psi(s)\,ds \quad \text{with the}$$

natural b.c. $v'(t)=0$ and $v(0)=0$. $(\psi(0)=0)$

So our Euler-Equation is $1 + v''(\tau) = 0$

$v(0) = 0$

$v'(t) = 0$

$v'(\tau) = -\tau + t \Rightarrow$, $v^*(\tau) = \frac{-\tau^2}{2} + \tau$ but we want the sup.

You do indeed get $\frac{t^3}{6}$ as was desired. Another example

Recall $P\{\sup_{0\le s\le t} \beta(s) \le \alpha\} = \sqrt{\frac{2}{\pi t}}\int_0^\alpha e^{-\frac{u^2}{2t}}\,du$ consider the expected value for Brownian motion $E\{e^{\sup_{0\le s\le t}\beta(s)}\} = \int_0^\infty e^\alpha\,dP\{\ \le\alpha\}$

$$= \int_0^\infty e^\alpha \sqrt{\frac{2}{\pi t}}\, e^{-\frac{\alpha^2}{2t}}\,d\alpha$$

Δ variable

$$= \sqrt{\frac{2}{\pi t}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{\alpha}{\sqrt{t}}-\sqrt{t}\right)^2}\,d\alpha\, e^{t/2} = e^{\frac{t}{2}}\sqrt{\frac{2}{\pi}}\int_{-\sqrt{t}}^\infty e^{-u^2/2}\,du$$

$$\therefore \lim \frac{1}{t}\log E\{e^{\sup_{0\le s\le t}\beta(s)}\} = \frac{1}{2}.$$

Brownian Scaling: $\beta(\tau)$ is $N(0,\tau)$
$\beta(c\tau)$ is $N(0,c\tau)$
$\sqrt{c}\,\beta(\tau)$ is $N(0,c\tau)$

$$E\{\beta(\tau)\beta(s)\} = \min(\tau,s)$$

$E\{\beta(c\tau)\beta(cs)\} = c\min(\tau,s)$ we claim that we can pull the c out so that $E\{\beta(c\tau)\beta(cs)\} = E\{\sqrt{c}\,\beta(\tau)\sqrt{c}\,\beta(s)\} = c\min(\tau,s)$

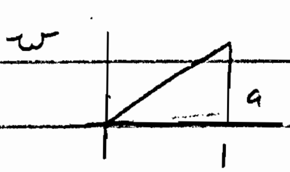
Now:

$$E\{e^{\sup_{0\le s\le t}\beta(s)}\} = E\{e^{\sup_{0\le\tau\le 1}\beta(t\tau)}\} = E\{e^{\sup_{0\le\tau\le 1}\sqrt{t}\,\beta(\tau)}\}$$

$$= E\{e^{t\sup_{0\le\tau\le 1}\frac{1}{\sqrt{t}}\beta(\tau)}\} \quad \text{let } t = \tfrac{1}{\varepsilon}$$

$$= E\{e^{\frac{1}{\varepsilon}\sup_{0\le\tau\le 1}\sqrt{\varepsilon}\,\beta(\tau)}\}$$ and so with our knowledge

$$\lim_{t\to\infty}\frac{1}{t}\log E\{e^{\sup_{0\le s\le t}\beta(s)}\} = \lim_{\varepsilon\to 0}\varepsilon\log E\{e^{\frac{1}{\varepsilon}\sup_{0\le\tau\le 1}\sqrt{\varepsilon}\,\beta(\tau)}\}$$



$$= \sup_{w\in C_0^*(0,1)}\left[\sup_{0\le\tau\le 1} w(\tau) - \frac{1}{2}\int_0^1 [w'(\tau)]^2\,d\tau\right]$$

$$= \max_{a>0}\left[a - \frac{a^2}{2}\right] = \frac{1}{2}$$ when $a=1$ is the ma

We will eventually prove: Let Ω be a seperable metric space and let $\mathcal{B}$ be the Borel sets in Ω. For any $\varepsilon>0$ let P^ε be a probability measure on $(\Omega,\mathcal{B})$. Let $A:\Omega\to[0,\infty]$ satisfying

1) for every closed set $C\in\Omega$ we want

$$\overline{\lim_{\varepsilon\to 0}}\ \varepsilon\log P^\varepsilon(C) \le -\inf_{w\in C} A[w]$$

2) for every open set $G \in \Omega$ we also want

$$\underline{\lim_{\varepsilon \to 0}} \, \varepsilon \log P^{\varepsilon}(G) \geq -\inf_{w \in G} A[w]$$

Then for any $F: \Omega \to \mathbb{R}$ which is bounded and continuous

$$\lim_{\varepsilon \to 0} \varepsilon \log E^{P^{\varepsilon}}\left\{ e^{\frac{1}{\varepsilon} F[w]} \right\} = \sup_{w \in \Omega} \left[F[w] - A[w] \right].$$

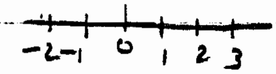
Brownian motion scaling gives $E\left\{ e^{\frac{1}{\varepsilon} F[\sqrt{\varepsilon}\, w]} \right\}$.

12/14

Prob.

Let $\Omega = [0,1]$ with ± 1 binary expansion and

$S_n = X_1 + X_2 + \cdots + X_n$ and a random walk



S_n is the length of the random walk.

and $P\left\{ \overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n \log\log n}} = 1 \right\} = 1$

$P\left\{ \underline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n \log\log n}} = -1 \right\} = 1$, $\frac{S_n}{n} \to 0$ for a.e. w and

also

$$P\left\{ a < \frac{S_n}{\sqrt{n}} < b \right\} = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du.$$

<u>Theorem</u>: (L.I.L. Khintchine). Let $X_1, X_2, \ldots$ be i.i.d. Bernouli r.v.

$P\{X_j = 1\} = p$

$P\{X_j = 0\} = q = 1-p$ and $S_n = \sum_{j=1}^n X_j$ then

(1) $P\left\{ \overline{\lim_{n\to\infty}} \frac{S_n - np}{\sqrt{2n \log\log n}} = 1 \right\} = 1$; (2) $P\left\{ \underline{\lim_{n\to\infty}} \frac{S_n - np}{\sqrt{2n \log\log n}} = -1 \right\} = 1$

Recall the <u>Laplace-De Moivre theorem</u>: For fixed x

$$\lim_{n\to\infty} P\left\{ \frac{S_n - np}{\sqrt{npq}} > x \right\} = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2}\, du \quad \text{or}$$

$$\lim_{n\to\infty} \frac{P\left\{ \frac{S_n - np}{\sqrt{npq}} > x \right\}}{\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2}\, dx} = 1.$$ Does this hold if $x = x(n)$.

Yes if, $\lim_{n\to\infty} \frac{x^3(n)}{\sqrt{n}} = 0$. This is a fact. Now the proof.

<u>Proof</u>: To prove (1) we must show

(3) If $\lambda > 1$ then for a.e. w $\exists N \ni \forall n > N$ implies

$S_n \le np + \lambda\sqrt{2n\log\log n}$. And

(4) If $\lambda < 1$ then for almost all ω there are infinitely many n for which $S_n > np + \lambda\sqrt{2n\log\log n}$.

Remark: $\exists$ a constant $\delta > 0$ depending on p but independent of $n \ni$

$$P\{S_n > np\} > \delta \quad \forall n$$

Proof: $P\{S_n > np\} = \sum_{np < k \le n} \binom{n}{k} p^k q^{n-k} > 0$ and

$$\lim_{n\to\infty} P\left\{\frac{S_n - np}{\sqrt{npq}} > 0\right\} = \frac{1}{2}.)$$

Lemma: $P\{\max_{1\le k\le n}(S_k - kp) > x\} \le \frac{1}{\delta} P\{S_n - np > x\}$

Proof. $P\{\max_{1\le k\le n}(S_k - kp) > x\} = P\{S_1 - p > x\} + \sum_{r=2}^{n} P\{S_r - rp > x;\ S_j - jp \le x,\ 1 \le j \le r-1\}$

Consider $\{\omega : S_1 - p > 0,\ S_n - S_1 > (n-1)p\}$

$$\bigcup_{r=2}^{n-1} \{\omega : S_r - rp > x;\ S_j - jp > x;\ 1 \le j \le r-1;\ S_n - S_r > (n-r)p\}$$

$$\cup \{\omega : S_n - np > x;\ S_j - jp \le x;\ 1 \le j \le n-1\} \subset \{\omega : S_n - np > x\}$$

These are mutually disjoint on the left, because of the staggered inequalities. Also we can factor the probabilities. So

$$P\{S_1 - p > x\}P\{S_n - S_1 > (n-1)p\} + \sum_{r=2}^{n-1} P\{S_r - rp > x : \underset{1\le j\le r-1}{S_j - jp \le x}\}P\{S_n - S_r > (n-r)p\}$$

$$+ P\{S_n - np > x;\ S_j - jp \le x\} \le P\{S_n - np > x\}$$

So $\delta P\{\max_{1\le k\le n}(S_k - kp) > x\} \le P\{S_n - np > x\}$. Q.E.D.

Prob.

To prove (3) let $\lambda > 1$. Choose $\gamma \ni \lambda > \gamma > 1$. And let $n_r = [\gamma^r]$ $r = 1, 2, 3, \cdots$ (greatest integer). Let

$$B_r = \{ w : \max_{n_r \le n \le n_{r+1}} (S_n - np) > \lambda \sqrt{2 n_r pq \log\log n_r} \}.$$

We show that $\sum_{r=1}^{\infty} P\{B_r\} < \infty$ and so by the first Borel-Cantelli lemma for almost all w only finitely many of the events B_r can occur. Note that this will imply (3) because of the construction of B_r. This means

$P\{\limsup B_r\} = 0$ or for almost all w at most a finite number of the B_r occur. So for almost all w $\exists R \ni \forall r \ge R$

$$\max_{n_r \le n \le n_{r+1}} (S_n - np) \le \lambda \sqrt{2 n_r pq \log\log n} \le \lambda \sqrt{2 npq \log\log n}.$$

To see that $\sum_{r=1}^{\infty} P(B_r) < \infty$: From the lemma

$$P(B_r) \le \frac{1}{\delta} P\{ S_{n_{r+1}} - n_{r+1} p > \lambda \sqrt{2 n_r pq \log\log n_r} \}$$ which

is $$P\{B_r\} \le \frac{1}{\delta} P\{ \frac{S_{n_{r+1}} - n_{r+1} p}{\sqrt{n_{r+1} pq}} > \lambda \sqrt{\frac{2 n_r}{n_{r+1}} \log\log n_r} \}$$

But this last Probability is asymptotic to

$$\sim \frac{1}{\sqrt{2\pi}} \int_{x_r}^{\infty} e^{-u^2/2} du \quad \text{where } x_r = \lambda \sqrt{2 \frac{n_r}{n_{r+1}} \log\log n_r}$$

$$\sim \frac{1}{x_r \sqrt{2\pi}} e^{-x_r^2/2}$$ as $r \to \infty$. So what

is $\frac{n_r}{n_{r+1}} \sim \gamma^r$ and

$\frac{1}{\gamma} > \frac{1}{\lambda}$ so for large r

$P\{B_r\} \le \frac{1}{\delta} e^{-\lambda \log\log n_r} = \frac{1}{\delta}\frac{1}{(\log n_r)^\lambda} \sim \frac{1}{\delta}\frac{1}{(r\log\gamma)^\lambda}$. So by comparison to the geometric series, since $\lambda > 1$ then $\sum_{r=1}^{\infty} P\{B_r\} < \infty$, and we have the case (3).

Now to prove (4) we will use the Second Borel-Cantelli Lemma. Let $\lambda < 1$, choose $\lambda < \eta < 1$ and so close to 1 that $1-\eta < \left(\frac{\eta-\lambda}{2}\right)$, then choose γ an integer $\ni \frac{\gamma-1}{\gamma} > \eta$. Let $n_r = \gamma^r$ $r = 1,2,3,\dots$ and let $A_r = \{w : (S_{n_r} - S_{n_{r-1}}) - (n_r - n_{r-1})p > \eta\sqrt{2n_r \log\log n_r}\}$ $r = 1,2,3,\dots$ Note that the A_r are independent. So if we can show that

$\sum_{r=1}^{\infty} P(A_r) = \infty$ we have by the second Borel-Cantelli lemma that

$P\{\limsup_{r\to\infty} A_r\} = 1$, i.e. for almost all w there are infinitely many r such that $(S_{n_r} - S_{n_{r-1}}) - (n_r - n_{r-1})p >$

$$(*) \quad \eta\sqrt{2n_r pq \log\log n_r}.$$

From the first part of the proof with $\lambda = 2$ we have that for almost all w, $\exists N \ni n \ge N \Rightarrow$

$$S_n \ge np - 2\sqrt{2npq \log\log n}$$ and in particular there exists a R so that for $r \ge R \Rightarrow$

$$(**) \quad S_{n_{r-1}} \ge n_{r-1}p - 2\sqrt{2n_{r-1}pq \log\log n_{r-1}}$$

Hence for almost w $\exists$ infinitely many rs $\ni$ both inequalities (*) and (**) hold. Adding them we have for almost all w $\exists$ infinitely many $r \ni$

$S_{n_r} - n_r p >$ (This in brackets is to be ignored.)

i.e. (**)$^+$ $S_{n_{r-1}} - n_{r-1}p > -(\eta-\lambda)\sqrt{2pqn_r \log\log n_r}$ because $n_r > n_{r-1}$ and $4n_{r-1} = \frac{4n_r}{\gamma} < 4n_r(1-\eta)$ but $1-\eta < (\eta-\lambda)^2 n_r$ so we may replace $4n_{r-1}$ and get this. Now add (*) and (**)$^+$. For almost all ϖ there are infinitely many $r \ni$

$$S_{n_r} - n_r p > \lambda\sqrt{2pqn_r \log\log n_r}$$

and so we have (4) and the theorem since this is a particular set of infinitely many r's. But we must consider:

$$P\{A_r\} = P\left\{ \frac{S_{n_r} - S_{n_{r-1}} - (n_r - n_{r-1})p}{\sqrt{(n_r - n_{r-1})pq}} > \eta\sqrt{\frac{2n_r \log\log n_r}{(n_r - n_{r-1})}} \right\}$$

but this is asymptotic

$$\sim \frac{1}{\sqrt{2\pi}} e^{-x_r^2/2} \quad \text{where} \quad x_r = \eta\sqrt{\frac{2n_r}{(n_r - n_{r-1})}\log\log n_r}.$$

<u>Brownian</u> <u>Motion</u>: <u>Theorem</u>: Let Ω be a seperable metric space and let $\mathcal{B}$ be the Borel sets of Ω for any $\varepsilon > 0$ let P^ε be a probability measure on $(\Omega, \mathcal{B})$. Let $A: \Omega \to [0,\infty]$ be any functional satisfying

i) For any closed set $C \subset \Omega$

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log P^\varepsilon(C) \leq -\inf_{\varpi\in C} A[\varpi]$$

ii) For any open set $G \subset \Omega$

$$\underline{\lim}\, \varepsilon \log P^\varepsilon(G) \geq -\inf A[\ldots]$$

So $\lim_{\varepsilon\to 0} u(x,t;\varepsilon) = u_0(b) = \frac{x-b}{t}$ provided there is a unique b.

Suppose V where not a given function and solve u as a function of V. Then we would get an integral equation in function space.

Theorem:

If $V(y)\to\infty$ as $|y|\to\infty$ so that the eigenvalue problem

$$\frac{1}{2}\varphi''(y) - V(y)\varphi(y) = -\lambda\varphi(y)$$ has a discrete spectrum.

Then the lowest eigenvalue is given by

$$\lim_{t\to\infty}\frac{1}{t}\log E\left\{e^{-\int_0^t V(\beta(s))ds}\right\} = -\lambda_1.$$

Proof: Let $u(x,t) = E_x\left\{e^{-\int_0^t V(\beta(s))ds}\right\}$. Then by Feynman-Kac

$u(x,t)$ solves
$$u_t = \frac{1}{2}u_{xx} - V(x)u$$
$$u(x,0)=1$$

On the other hand, by separating variables

$$u(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(x)\int_{-\infty}^{\infty}\varphi_j(y)\,dy$$ where

$\lambda_1,\lambda_2,\dots;\ \varphi_1,\varphi_2,\dots;$ are the eigenvalues and normalized eigenfunct of

$$\frac{1}{2}\varphi'' - V\varphi = -\lambda\varphi.$$

So $\lim_{t\to\infty}\frac{1}{t}\log u(x,t) = -\lambda_1$ Child's play. But also by Rayleigh-Ritz

$$\lambda_1 = \inf_{\varphi,\ \|\varphi\|_{L^2}=1}\left\{\int_{-\infty}^{\infty}V(y)\varphi^2(y)dy + \frac{1}{2}\int_{-\infty}^{\infty}[\varphi'(y)]^2dy\right\}.$$

Prob. ⑥

$F: \Omega \to \mathbb{R}$ which is bounded and continuous such

$$\lim_{\varepsilon \to 0} \varepsilon \log E^{P^\varepsilon}\{e^{\frac{1}{\varepsilon}F(w)}\} = \sup_{w \in \Omega} [F[w] - A[w]]$$

$|\le M$ for all $w \in \Omega$. For any integer N and $, \dots, N-1$ define

$$C_j^N = \{w \in \Omega : \tfrac{jM}{N} \le F(w) \le \tfrac{(j+1)M}{N}\}$$

ntinuous and C_j^N is a closed set in Ω from i)

$\varepsilon P^\varepsilon(C_j^N) \le -\inf_{w \in C_j^N} A[w]$. Now

$$\} \le \sum_{j=-N}^{N} P^\varepsilon(C_j^N)\, e^{\frac{1}{\varepsilon}\left(\frac{(j+1)M}{N}\right)} \quad \text{therefore}$$

$$E^{P^\varepsilon}\{e^{\frac{1}{\varepsilon}F(w)}\} \le \max_{-N \le j \le N-1} \left(\frac{(j+1)M}{N} - \inf_{w \in C_j^N} A[w]\right)$$

$$= \frac{M}{N} + \max_{-N \le j \le N-1} \left(\frac{jM}{N} - \inf_{w \in C_j^N} A[w]\right)$$

$$+ \max_{-N \le j \le N-1} \left(\sup_{w \in C_j^N} \frac{jM}{N} - A[w]\right)$$

$$\max_{-N \le j \le N-1} \sup_{w \in C_j^N} \left(F[w] - A[w]\right) = \frac{M}{N} + \sup_{w \in \Omega} \left(F[w] - A[w]\right)$$

$$E^{P^\varepsilon}\{e^{-\frac{1}{\varepsilon}F(w)}\} \le \sup_{w \in \Omega} [F[w] - A[w]].\ \text{So}$$

$w_1 \in \Omega$

ood of $w_1 \Rightarrow$

because F i

integrating on G only

by

$A[w]$

is arbitrary

$[w])$ and so

of the

will be

$,1)$:

$\} \le \frac{\varepsilon}{\varepsilon}\}$

And the A function will be $A[w] = \begin{cases} \frac{1}{2}\int_0^1 [\beta'(s)]^2 ds & \text{if } \beta \in C_0^*(0,1) \\ \infty & \beta \notin C_0^*(0,1) \end{cases}$

$C_0^*(0,1)$ are β's which are absolutely continuous and have derivatives in L^2, i.e. $\int_0^1 [\beta'(s)]^2 ds < \infty$ for these β's. A subset of $C_0(0,1)$.

<u>Theorem</u>: Let $\beta(s)$ $0 \le s < \infty$ be B.M. on $[0,\infty)$ with $\beta(0)=0$.

then $$P\left\{ \overline{\lim_{t\to\infty}} \frac{\beta(t)}{\sqrt{2t\log\log t}} = 1 \right\} = 1$$

$$P\left\{ \underline{\lim_{t\to\infty}} \frac{\beta(t)}{\sqrt{2t\log\log t}} = -1 \right\} = 1$$ proved by Paul Levy. But

then Chung (1948) proved

$$P\left\{ \overline{\lim_{t\to\infty}} \frac{\max_{0\le s\le t} \beta(s)}{\sqrt{2t\log\log t}} = 1 \right\} = 1.$$

The Proof of the Law of the Iterated Logarithm.

$$A_r = \left\{ \frac{(S_{n_r} - S_{n_{r-1}}) - (n_r - n_{r-1})p}{\sqrt{(n_r - n_{r-1})pq}} > \eta \sqrt{2\left(\frac{n_r}{n_r - n_{r-1}}\right) \log\log n_r} \right\}$$

$n_r = \gamma^r$, $\eta < 1$ $\frac{\gamma - 1}{\gamma} > \eta$. We need to show that $\sum_{r=0}^{\infty} P(A_r)$ diverges. $\frac{n_r}{n_r - n_{r-1}} = \frac{\gamma}{\gamma - h}, < \frac{1}{\eta}$

$$P\{A_r\} > P\left\{ \text{———} > \sqrt{2\eta \log\log n_r} \right\}$$ using the asymptotic formu

$$> \frac{1}{2\eta \log\log n_r} e^{-\eta \log\log n_r}$$

$$= \frac{1}{2\eta \log\log n_r (\log n_r)^{\eta}} = \frac{1}{2\eta \log(r\log\gamma)(r\log\gamma)^{\eta}} > \frac{1}{r}$$

which $\sum_{r=0}^{\infty} \frac{1}{r}$ diverges, and so by the second Borel-Cantelli lemma we prove it.

Back to Brownian motion:

$$E\left\{ e^{\frac{1}{\varepsilon} F[\sqrt{\varepsilon}\beta(\cdot)]} \right\}$$ β is B.M. on $0 \le s \le t$, $\beta(0) = 0$

Then $$\lim_{\varepsilon \to 0} \varepsilon \log E\left\{ e^{\frac{1}{\varepsilon} F[\sqrt{\varepsilon}\beta(\cdot)]} \right\} = \sup_{w \in C_0^*(0,t)} \left[F[w] - A[w] \right]$$

$$A[w] = \frac{1}{2} \int_0^t [w'(s)]^2 ds$$

Prob.

onsider $\lim_{\varepsilon \to 0} \frac{E\{G[\sqrt{\varepsilon}\beta(\cdot)]e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta(\cdot)]}\}}{E\{e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta(\cdot)]}\}}$ "="

$$\frac{\int G(\sqrt{\varepsilon}\beta(\cdot))e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta(\cdot)] - \frac{1}{2}\int_0^t [\beta'(s)]^2 ds}\,\delta\beta}{\int e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta(\cdot)] - \frac{1}{2}\int_0^t [\beta'(s)]^2 ds}\,\delta\beta} =$$ change of variables

$$= \frac{\int G(x(\cdot))e^{\frac{1}{\varepsilon}[F[x(\cdot)] - \frac{1}{2}\int_0^t [x'(s)]^2 ds]}\,\delta x}{\int e^{\frac{1}{\varepsilon}[F[x(\cdot)] - \frac{1}{2}\int_0^t [x'(s)]^2 ds]}\,\delta x}$$ as $\varepsilon \to 0$ this Goes to the δ function kind of

$= G[w^*(\cdot)]$ where w^* maximizes $\sup_{w \in C_0^*(0,t)}[F[w] - A[w]]$, provided that things are unique. In many applied problems this situation actually occurs. If F depends on α then sometimes the $w_\alpha^*(\cdot)$ will vary not nicely on α. Sometimes this is because there is non-uniqueness as a function of α.

Burger's Equation: $u_t + uu_x = \frac{\varepsilon}{2}u_{xx}$ $\quad -\infty < x < \infty$

$u(x,0) = u_0(x)$ $\quad t > 0$

and $\int_0^x u_0(\eta)d\eta = o(x^2)$ as $|x| \to \infty$. This mimics fluid dynamics.

Hopf discovered $v_t = \frac{\varepsilon}{2}v_{xx}$ $\quad v(x,0) = e^{-\frac{1}{\varepsilon}\int_0^x u_0(\eta)d\eta}$ and

if $u = -\varepsilon\frac{v_x}{v}$ then we transform the equation. So

$$V(x,t,\varepsilon) = \frac{1}{\sqrt{2\pi t\varepsilon}} \int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\int_0^y u_0(\eta)d\eta} \, e^{-\frac{(x-y)^2}{2\varepsilon t}} dy$$

nd hence
$$u(x,t,\varepsilon) = \frac{\int_{-\infty}^{\infty} \left(\frac{x-y}{t}\right) e^{-\frac{1}{\varepsilon}\left\{\int_0^y u_0(\eta)d\eta + \frac{(y-x)^2}{2t}\right\}} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\left[\int_0^y u_0(\eta)d\eta + \frac{(y-x)^2}{2t}\right]} dy}$$

et $F(y) = \int_0^y u_0(\eta)d\eta + \frac{(y-x)^2}{2t}$ where $F(y)$ is cont. on $-\infty<y<\infty$
nd so
$$\lim_{|y|\to\infty} \frac{F(y)}{y^2} = \frac{1}{2t} \quad \therefore F(y) \text{ has a minimum for each } (x,t).$$

erhaps there exists more than one point y at which the minimum is
chieved. Hopf showed that If at a point (x,t) $\exists$ a single value
f y which minimizes $F(y)$, call this y, b. Then

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = \left(\frac{x-b}{t}\right) = u_0(b).$$

Consider $u_t + uu_x = \frac{\varepsilon}{2} u_{xx} + V'(x)$, $u(x,0) = u_0(x)$, $-\infty<x<\infty$, $t>0$

and $\int_0^x u_0(\eta)d\eta = o(x^2)$ as $|x|\to\infty$. Try the Hopf-
ole transformation.

$$V_t = \frac{\varepsilon}{2} V_{xx} - \frac{1}{\varepsilon} V(x) V$$

$$V(x,0) = e^{-\frac{1}{\varepsilon}\int_0^x u_0(\eta)d\eta} \quad \text{which is lovely.}$$

So writedown the Feynman-Kac formula:

$$V(x,t;\varepsilon) = E_x\left\{ e^{-\frac{1}{\varepsilon}\int_0^t V(\sqrt{\varepsilon}\beta(s))ds - \frac{1}{\varepsilon}\int_0^{\sqrt{\varepsilon}\beta(t)} u_0(\eta)d\eta} \right\} =$$

12/21

Prob. ④

$$= E_0\left\{ e^{-\frac{1}{\varepsilon}\left[\int_0^t V(\sqrt{\varepsilon}\beta(s)+x)\,ds + \int_0^{\sqrt{\varepsilon}\beta(t)+x} u_0(\xi)\,d\xi\right]}\right\}$$

so inverting to get $u(x,t;\varepsilon)$

$$u(x,t;\varepsilon) = \frac{E_0\left\{\left[\int_0^t V'(\sqrt{\varepsilon}\beta(s)+x)\,ds + u_0(\sqrt{\varepsilon}\beta(t)+x)\right] e^{-\frac{1}{\varepsilon}\int_0^t V(\sqrt{\varepsilon}\beta(s)+x)\,ds - \frac{1}{\varepsilon}\int_0^{\sqrt{\varepsilon}\beta(t)+x} u_0(\xi)\,d\xi}\right\}}{E_0\left\{e^{-\frac{1}{\varepsilon}\int_0^t V(\sqrt{\varepsilon}\beta(s)+x)\,ds - \frac{1}{\varepsilon}\int_0^{\sqrt{\varepsilon}\beta(t)+x} u_0(\xi)\,d\xi}\right\}}$$

Let $F[\beta(\cdot)] = \int_0^t V(\beta(s)+x)\,ds + \int_0^{\beta(t)+x} u_0(\xi)\,d\xi$

$G[\beta(\cdot)] = \int_0^t V'(\beta(s)+x)\,ds + u_0(\beta(t)+x)$. Then

$$u(x,t;\varepsilon) = \frac{E\left\{G[\sqrt{\varepsilon}\beta(\cdot)]\, e^{-\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta(\cdot)]}\right\}}{E\left\{e^{-\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta(\cdot)]}\right\}}$$

$\lim_{\varepsilon\to 0} u(x,t;\varepsilon) = G[w^*]$ provided $w^*(\cdot)$ is the unique function which minimizes $\inf_{w\in C_0^*(0,t)} [F[w] + A[w]]$

i.e. if at a point (x,t) $\exists$ such a unique minimizing function w^*, then the limit exists and is.

$$G[w^*(\cdot)] = \int_0^t V'(w^*(s)+x)\,ds + u_0(w^*(t)+x).$$

Now consider the variational problem:

$$\inf_{w\in C_0^*(0,t)} \left[\int_0^t V(w(s)+x)\,ds + \int_0^{w(t)+x} u_0(\xi)\,d\xi + \frac{1}{2}\int_0^t [w'(s)]^2\,ds\right]$$

$\shortparallel$

And consider the Frechet Derivative

$$\delta H\Big|_{\psi} = \frac{dH[w+h\psi]}{dh}\Big|_{h=0} = \int_0^t V'(w(s)+x)\psi(s)\,ds$$

$$+ u_0(w(t)+x)\psi(t) + w'(t)\psi(t) - \int_0^t w''(s)\psi(s)\,ds$$

$$\frac{1}{2}\int_0^t [w'(s)]^2 ds \Rightarrow \frac{d}{dh}\Big|_{h=0} \frac{1}{2}\int_0^t [w'(s)+h\psi(s)]^2 ds = \int_0^t w'(s)\psi'(s)\,ds$$ integration by parts

$$= w'(t)\psi(t) - \int_0^t w''(s)\psi(s)\,ds$$

So we must have $V'(w(s)+x) = w''(s)$ $\quad 0 \le s \le t$

$w(0) = 0$ since $w \in C_0^*(0,t)$

$w'(t) = -u_0(w(t)+x)$ to get rid of the other stuff.

Now try this out on Hopf's result: Suppose $V \equiv 0$. Then

$w''(s) = 0$, $w(0) = 0$

$w'(t) = -u_0(w(t)+x)$

So $w(s) = cs$ for some constant c

$w'(t) = c = -u_0(ct+x)$ Let $c = \frac{b-x}{t}$

$\frac{b-x}{t} = -u_0(b)$ or $u_0(b) = \frac{x-b}{t}$. So is there a unique b that satisfies this and then we get a unique $w^*(s)$. So $w^*(s) = \left(\frac{b-x}{t}\right)s$ when there is uniqueness.

Therefore $\lim_{t\to\infty} \frac{1}{t} \log E\left\{ e^{-\int_0^t V(\beta(s))ds} \right\} =$

$$-\inf_{\substack{\psi \in L^2 \\ \|\psi\|=1}} \left[\int_{-\infty}^{\infty} V(y)\, \psi^2(y)\, dy + \frac{1}{2} \int_{-\infty}^{\infty} [\psi'(y)]^2 dy \right].$$

1/25

Probability

Feynman-Kac:

Consider $u_t = \frac{1}{2} u_{xx} - V(x)u \qquad u(x,0) = u_0(x) \quad , t > 0$

$V(x)$ measurable and bounded below

The solution is given by $E_x\left\{ e^{-\int_0^t V(\beta(s))ds} u_0(\beta(t)) \right\}$ where $\beta(s)$ $0 \le s < \infty$ is B.M. with $\beta(0) = x$. That is

$$u(x,t) = \int_{-\infty}^{\infty} u_0(y) \underbrace{E_x\left\{ e^{-\int_0^t V(\beta(s))ds} \delta(\beta(t)-y) \right\}} dy.$$

We will show that

$\varphi(x,t,y)$ satisfies

$$\text{I} \begin{cases} \varphi_t = \frac{1}{2} \varphi_{xx} - V(x)\varphi \\ \varphi(x,0,y) = \delta(x-y) \end{cases} \quad \text{and also}$$

$$\text{II} \begin{cases} \varphi_t = \frac{1}{2} \varphi_{yy} - V(y)\varphi \\ \varphi(x,0,y) = \delta(y-x) \end{cases}$$

We prove II: We will say

$$\varphi(x,t,y) = E_x\left\{ e^{-\int_0^t V(\beta(s))ds} \delta(\beta(t)-y) \right\}.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu y} E_x\left\{ e^{-\int_0^t V(\beta(s))} e^{i\mu\beta(t)} \right\} d\mu.$$

Note:

$$e^{-\int_0^t V(\beta(s))ds} = 1 - \int_0^t V(\beta(\tau)) e^{-\int_0^\tau V(\beta(s))ds} d\tau. \text{ So}$$

$$\varphi(x,t,y) = E_x\{\delta(\beta(t)-y)\} - \int_0^t E_x\left\{ V(\beta(\tau)) e^{-\int_0^\tau V(\beta(s))ds} \right\} d\tau$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu y} \int_0^t E_x\left\{ V(\beta(\tau)) e^{-\int_0^t V(\beta(s))ds} e^{i\mu\beta(\tau)} e^{i\mu(\beta\ldots}\right.$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu y} \int_0^t E_x\left\{ V(\beta(\tau)) e^{-\int_0^t V(\beta(s))ds} e^{i\mu\beta(\tau)} \right\} e^{-\frac{\mu^2(t-\tau)}{2}} d\mu\, d\tau$$

indep. increme[nts]

Here with Markov process we use

$$\frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu y}\int_0^t\int_{-\infty}^{\infty} V(\xi)\, e^{i\mu\xi}\, \underbrace{E_x\left\{e^{-\int_0^\tau V(\beta(s))ds}\,\delta(\beta(\tau)-\xi)\right\}}_{\varphi(x,\tau,\xi)} e^{-\mu^2\frac{(t-\tau)}{2}}\, d\xi\, d\mu\, d\tau$$

$$\frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu y}\int_0^t\int_{-\infty}^{\infty} V(\xi)\, e^{i\mu\xi}\, \varphi(x,\tau,\xi)\, e^{-\mu^2\frac{(t-\tau)}{2}}\, d\xi\, d\mu\, d\tau$$

$$\varphi(x,t,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} - \int_{-\infty}^{\infty} d\xi \int_0^t d\tau\, V(\xi)\,\varphi(x,\tau,\xi)\, \frac{1}{\sqrt{2\pi(t-\tau)}}\, e^{-\frac{(y-x)^2}{2(t-\tau)}}$$ do the μ integral

This is an integral equation for φ and is equivalent (to II). Both sides

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial y^2}\right)\varphi(x,t,y) = 0 - V(y)\,\varphi(x,t,y)$$ with careful analysis!

$\varphi(x,0,y) = \delta(y-x)$. Now we have proved for II and we will show $\varphi(x,t,y) = \varphi(y,t,x)$.

$$\varphi(x,t,y) = E_x\left\{e^{-\int_0^t V(\beta(s))ds}\,\delta(\beta(t)-y)\right\}$$ translating to zero

$$= E_0\left\{e^{-\int_0^t V(\beta(s)+x)ds}\,\delta(\beta(t)+x-y)\right\}$$

$$= E\left\{e^{-\int_0^t V(\beta(s)+y-\beta(t))}\,\delta(\beta(t)+x-y)\right\}$$ δ function is symmetric

$$= E\left\{e^{-\int_0^t V(\beta(s)+y-\beta(t))}\,\delta(y-x-\beta(t))\right\}$$ replace $-\beta(t)$, $\beta(t)$

$\beta(s)-\beta(t)$, $\beta(s)-\beta(t)$

$$= E\left\{e^{-\int_0^t V(\beta(t)+y-\beta(s))ds}\,\delta(y-x+\beta(t))\right\}$$

$$= E_y\left\{e^{-\int_0^t V(\beta(t)-\beta(s))ds}\,\delta(\beta(t)-x)\right\}$$ Let $s = t-\tau$,

$$= E_y\left\{e^{-\int_0^t V(\beta(t)-\beta(t-\tau))d\tau}\,\delta(\beta(t)-x)\right\}$$ this is $\beta(\tau)$

$$= E_y\left\{e^{-\int_0^t V(\beta(\tau))d\tau}\,\delta(\beta(t)-x)\right\} = \varphi(y,t,x).$$

le can also write: $u(x,t) = E_0\left\{ e^{-\int_0^t V(\beta(s)+x-\beta(t))\,ds} u_0(x-\beta(t)) \right\}$ this
: often seen that way.
f $V(x) = -x^4$ then $E_x\left\{ e^{+\int_0^t \beta^4(s)\,ds} u_0(\beta(t)) \right\}$ doesn't exist.

Nith Markov process we get $\varphi_t = L\varphi - V(x)\varphi$. L need not be
·lliptic, local or differential operator.
 review: $u_t = \frac{1}{2}u_{xx}$ $u(x,0,y) = \delta(x-y)$ $\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} = p(t,x,y)$

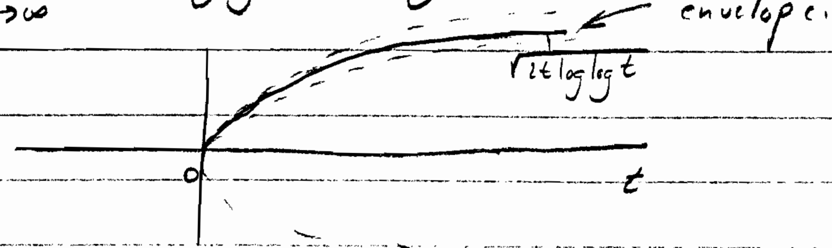
Suppose for another operator we get a fundamental solution $p(t,x,y)$
and we build up the measure with this $p(t,x,y)$. You then
get a Feynman-Kac like formula and it goes on. L is the
infinitesimal generator of a semi-group.

<u>Theorem</u>: (LIL for Brownian Motion): Let $\beta(s)$, $0 \le s < \infty$
$\beta(0)=0$ be B.M.,

(1) $P\left\{ \overline{\lim_{t\to\infty}} \frac{\beta(t)}{\sqrt{2t\log\log t}} = 1 \right\} = 1$ symmetrically

(2) $P\left\{ \underline{\lim_{t\to\infty}} \frac{\beta(t)}{\sqrt{2t\log\log t}} = -1 \right\} = 1$.

A picture



We prove (1), we must show:
a) If $c>1$ then for almost all paths there exists $T \ni$
$\forall t > T$ $\beta(t) < c\sqrt{2t\log\log t}$
b) If $c<1$ the for almost all paths there exists t arbitrarily la.
so: $\beta(t) > c\sqrt{2t\log\log t}$.

Proof: a) Let $c>1$, choose $q \ni 1<q<c^2$ and let $t_n = q^n$. Let $M_n = \sup_{t \le t_n} \beta(t)$. We recall that we have proved

$$P\left\{\sup_{0\le s\le t} \beta(t) \le \alpha\right\} = \begin{cases} \sqrt{\frac{2}{\pi}} \int_0^{\frac{\alpha}{\sqrt{t}}} e^{-\frac{u^2}{2}} du, & \alpha > 0 \\ 0 & \alpha \le 0 \end{cases}$$

Let $\alpha_n = P\{M_n \ge c\sqrt{2t_{n-1}\log\log t_{n-1}}\}$

$$P\{M_n \ge x\sqrt{t_n}\} = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2} du \le \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{u}{x} e^{-\frac{u^2}{2}} du$$
$$= \sqrt{\frac{2}{\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}$$

So $\alpha_n < \sqrt{\frac{2}{\pi}} \frac{1}{x_n} e^{-\frac{x_n^2}{2}}$ $\qquad x_n = c\sqrt{2\frac{t_{n-1}}{t_n}\log\log t_{n-1}}$
$$= c\sqrt{\frac{2}{q}\log(n-1)\log q}$$

So $\alpha_n < \sqrt{\frac{2}{\pi}} \frac{1}{x_n} [(n-1)\log q]^{-\frac{c^2}{q}}$ $\qquad x_n \sim k\sqrt{\log n}$

So $\alpha_n < (\qquad) \sim \frac{k}{\sqrt{\log n}} n^{-\frac{c^2}{q}}$ but $\frac{c^2}{q} > 1$

So $\sum_{n=1}^\infty \alpha_n < \infty$. By the first Borel-Cantelli lemma $\exists N \ni n > N$

$\Rightarrow M_n < c\sqrt{2t_{n-1}\log\log t_{n-1}}$. Let $T = t_N$ then for $t > T$ then

$t_{n-1} < t < t_n$ with $n > N$ so we have that for almost all paths $\exists T \ni t > T$ $\beta(t) \le M_n < c\sqrt{2t_{n-1}\log\log t_{n-1}}$
$< c\sqrt{2t\log\log t}$

b) Let $c<1$ and choose c' so that $c<c'<1$ and q so large that $c'\sqrt{\frac{q-1}{q}} - \frac{2}{\sqrt{q}} > c$ and fix q and let $t_n = q^n$. We must have $q>1$ also. Let $\beta_n = \beta(t_n) - \beta(t_{n-1})$ and let $\gamma_n = P\{\beta_n > x_n\sqrt{(\frac{q-1}{q})t_n}\}$.

[N]ote β_n is $N(0, t_n - t_{n-1}) = N(0, q^{n-1}(q-1)) = N\left(0, \left(\frac{q-1}{q}\right)t_n\right)$. Where

[x]$_n = c'\sqrt{2\log\log t_n} = c'\sqrt{2\log(n\log q)} \sim c'\sqrt{2\log n}$

[S]o $\quad \gamma_n = \frac{1}{\sqrt{2\pi}}\int_{x_n}^{\infty} e^{-\frac{u^2}{2}}\,du \sim \frac{1}{x_n\sqrt{2\pi}}\, e^{-\frac{x_n^2}{2}}$ as $x_n \to \infty$. Therefore

$\gamma_n \sim \frac{1}{x_n\sqrt{2\pi}}\,(n\log q)^{-(c')^2} \sim k\,\frac{1}{\sqrt{\log n}}\,(n\log q)^{-c'^2}$ but $c' < 1$ so

[t]he series $\sum_{n=0}^{\infty}\gamma_n$ diverges by comparison. Since the β_n's are indep. we have from the second Borel-Cantelli Lemma for almost all paths $\beta_n = \beta(t_n) - \beta(t_{n-1}) > x_n\sqrt{\left(\frac{q-1}{q}\right)t_n} = c'\sqrt{2\left(\frac{q-1}{q}\right)t_n\log\log t_n}$ for [i]nfinitely many n. From part one of this proof with $c = 2$ for almost all paths $\exists N \ni n > N \Rightarrow |\beta(t_{n-1})| < 2\sqrt{2t_{n-1}\log\log t_{n-1}}$

$$< 2\sqrt{\frac{2t_n}{q}\log\log t_n}$$

Combining the two, for almost all paths infinitely often (there exist infinitely many n) such that

$$\beta(t_n) \geq \beta(t_n) - \beta(t_{n-1}) - |\beta(t_{n-1})|$$
$$\geq c'\sqrt{2\left(\frac{q-1}{q}\right)t_n\log\log n} - 2\sqrt{\frac{2t_n}{q}\log\log t_n}$$

$$= \left(c'\sqrt{\frac{q-1}{q}} - \frac{2}{\sqrt{q}}\right)\sqrt{2t_n\log\log t_n} > c\sqrt{2t_n\log\log t_n}$$

So we have proved b). What is coming:

1. Action asymptotics proof (few weeks)
2. Strassen's version of L.I.L.

Review of Action Asymptotics: $C_0(0,1)$; $\beta(s)$ $0 \leq s \leq 1$ $\beta(0) = 0$.
Let P be B.M. measure on $C_0(0,1)$ the generator is $\frac{1}{2}\frac{\partial^2}{\partial x^2}$. Let P^ε be B.M. measure on $C_0(0,1)$ with the generator $\frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}$. This is the same except we multiply all the paths by $\sqrt{\varepsilon}$.

$$P^\varepsilon\left\{\int^1 \beta'^2 \ldots \frac{3}{\ldots}\right\} = P\left\{\int^1 \beta'^2 \ldots \frac{3}{\ldots}\right\}$$

From formal manipulations of flat integral we were led to guess that

$$\lim_{\varepsilon\to 0} \varepsilon \log E^{P^\varepsilon}\left\{e^{\frac{1}{\varepsilon}F[\beta]}\right\} = \sup_{w \in C_0^*(0,1) = C_0(0,1) \cap \{\text{absolutely continuous with derivatives in } L^2\}} \left[F[w] - \frac{1}{2}\int_0^1 [w'(s)]^2 ds\right]$$

We even tried out some examples and it worked. Then last semester towards the end we prove a theorem that said indeed:

If $F: C_0(0,1) \to \mathbb{R}$ is bounded and continuous the

$$\lim_{\varepsilon\to 0} \varepsilon \log E^{P^\varepsilon}\left\{e^{\frac{1}{\varepsilon}F[\beta]}\right\} = \sup_{w \in C_0^*(0,1)} \left[F[w] - \underbrace{\frac{1}{2}\int_0^1 [w'(s)]^2 ds}_{A[w]}\right]$$

Providing: 1) If C is a closed set in $C_0(0,1)$

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log P^\varepsilon(C) \le -\inf_{w\in C} A[w] \qquad \left(P^\varepsilon(C) \sim e^{-\frac{1}{\varepsilon}K}\right)$$

2) If G is an open set in $C_0(0,1)$ $\left(P^\varepsilon(G) \sim e^{-\frac{1}{\varepsilon}K'}\right)$.

$$\underline{\lim_{\varepsilon\to 0}}\, \varepsilon \log P^\varepsilon(G) \ge -\inf_{w\in G} A[w].$$

2/1

Prob.

__Theorem:__ If $C \subset C_0(0,1)$ is closed then

$$1)\ \overline{\lim_{\varepsilon\to\infty}}\ \varepsilon \log P^\varepsilon(C) \le -\inf_{w\in C} A[w]$$

and if $G \subset C_0(0,1)$ is open then $2)\ \underline{\lim_{\varepsilon\to 0}}\ \varepsilon \log P^\varepsilon(G) \ge -\inf_{w\in G} A[w]$

where $A: C_0(0,1) \to \mathbb{R}_+$ $A[\beta] = \begin{cases} \frac{1}{2}\int_0^1 [\beta'(s)]^2 ds & \beta \in C_0^*(0,1)\ \text{A.C.}\ \beta' \in L^2 \\ \infty & \text{otherwise} \end{cases}$

To prove the theorem we need some lemmas:

__1:__ Let $\beta_0 \in C_0(0,1)$ and for $\delta > 0$ let $S = S(\beta_0, \delta)$
$= \{ f \in C_0(0,1) \mid \|f - \beta_0\| < \delta \}$

then $3)\ \underline{\lim}\ \varepsilon \log P^\varepsilon(S(\beta_0,\delta)) \ge -A[\beta_0]$.

__Proof:__ If $\beta_0 \notin C_0^*(0,1)$ then $A[\beta_0] = \infty$ and there is nothing to prove so assume $\beta_0 \in C_0^*(0,1)$. Now $P^\varepsilon(S(\beta_0,\delta)) = P(S(\frac{\beta_0}{\sqrt{\varepsilon}}, \frac{\delta}{\sqrt{\varepsilon}}))$

So

$$P\{S(\tfrac{\beta_0}{\sqrt{\varepsilon}}, \tfrac{\delta}{\sqrt{\varepsilon}})\} = E^P\{\chi(\beta - \tfrac{\beta_0}{\sqrt{\varepsilon}}) \mid S(0, \tfrac{\delta}{\sqrt{\varepsilon}})\}$$

Cameron-Martin Translation Formula: $E^P\{G[\beta+\psi]\}$ $\psi \in C_0(0,1)$
$\psi' \in L^2$ $(\psi \in C_0^*(0,1))$

$$"=" \int G[\beta+\psi] e^{-\frac{1}{2}\int_0^1 [\beta'(s)]^2 ds}\, \delta\beta \qquad \beta + \psi = w$$

$$= \int G[w] e^{-\frac{1}{2}\int_0^1 [w'(s) - \psi'(s)]^2 ds}\, dw = e^{-\frac{1}{2}\int_0^1 \psi'(s)^2 ds} \int G[w] e^{\int_0^1 \psi'(s) dw(s)} e^{-\frac{1}{2}\int_0^1 [w'(s)]^2 ds} \delta w$$

So we guess $E^P\{G[\beta+\psi]\} = e^{-\frac{1}{2}\int_0^1 [\psi'(s)]^2 ds} E^P\{G[\beta] e^{\int_0^1 \psi'(s) d\beta(s)}\}$

So also we get $\int_0^1 \psi'(s)\, d\beta(s)$; $\psi(s) \in B.V.$ We could also consider a stochastic integral to give meaning to it. We will not prove it, however.

2/1 ②

Prob.

So $= e^{-\frac{1}{\varepsilon}A[\beta_0]} E^{P}\left\{ \chi(\beta)_{S(0,\frac{\delta}{\sqrt{\varepsilon}})} e^{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \beta_0'(s)\,d\beta(s)} \right\}$ and another trick

Jensen's Inequality: $E\{X^2\} \geq E\{X\}^2$

$E\{g(X)\} \geq g(E\{X\})$ if $g(y) = y^2$

g(y) is a convex function for any convex function.

From Jensen's Inequality; $g(y) = e^{-y}$

$$\frac{E^{P}\left\{ \chi(\beta)_{S(0,\frac{\delta}{\sqrt{\varepsilon}})} e^{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \beta_0'(s)\,d\beta(s)} \right\}}{P\left\{ S(0, \frac{\delta}{\sqrt{\varepsilon}}) \right\}}$$

$$= E^{P}\left\{ e^{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \beta_0'(s)\,d\beta(s)} \,\Big|\, S(0,\tfrac{\delta}{\sqrt{\varepsilon}}) \right\} \geq e^{-\frac{1}{\sqrt{\varepsilon}} \underbrace{E\left\{ \int_0^1 \beta_0'(s)\,d\beta(s) \,\big|\, S(0,\frac{\delta}{\sqrt{\varepsilon}}) \right\}}}$$

we have both β and $-\beta$ in our expectation is zero

This is greater than one

So $e^{-\frac{1}{\varepsilon}A[\beta_0]} E^{P}\left\{ \chi(\beta)_{S(0,\frac{\delta}{\sqrt{\varepsilon}})} e^{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \beta_0'(s)\,d\beta(s)} \right\}$

this is less than one

$$\geq e^{-\frac{1}{\varepsilon}A[\beta_0]} P\left\{ S(0, \tfrac{\delta}{\sqrt{\varepsilon}}) \right\}$$

$\varepsilon \log P^{\varepsilon}\{S(\beta_0, \delta)\} \geq -A[\beta_0] + \varepsilon \log P\{S(0, \frac{\delta}{\sqrt{\varepsilon}})\}$ therefore

goes to zero

$$\varliminf_{\varepsilon \to 0} \varepsilon \log P^{\varepsilon}\{S(\beta_0, \delta)\} \geq -A[\beta_0]$$

Now to prove ii) let G be an open set in $C_0(0,1)$ and $\beta_0 \in G$ then $\exists\, \delta > 0 \ni S(\beta_0, \delta) \subset G$ $\therefore$

$\varepsilon \log P^{\varepsilon}(G) \geq \varepsilon \log P^{\varepsilon}(S(\beta_0, \delta))$ and so from lemma just proved

$$\varliminf_{\varepsilon \to 0} \varepsilon \log P^{\varepsilon}(G) \geq -A[\beta_0]$$ since β_0 is any element of G we can say

$\underline{\lim}_{\varepsilon\to 0} \varepsilon P^{\varepsilon}(G) \geq -\inf_{\beta\in G} A[\beta]$. Now we prove 1). Lemma 2) Let $C \subset C_0(0,1)$ be closed and $\delta>0$. Let $C^{\delta} = \bigcup_{\beta\in C} S(\beta,\delta)$ the δ-sawage of C then

4) $\overline{\lim}\ \varepsilon \log P^{\varepsilon}(C) \leq -\inf_{w\in C^{\delta}} A[w]$. This is interesting.

Let n be a positive integer. For each $\beta\in C_0(0,1)$ let β_n be the polygonal function connecting $(0,0); (\frac{j}{n}, \beta(\frac{j}{n})); (1,\beta(1))$ clearly $\beta_n \in C_0(0,1)$. For every n and any $\delta>0$

$$5) \qquad P^{\varepsilon}(C) = P^{\varepsilon}\{\beta\in C : \|\beta-\beta_n\| \geq \delta\} + P^{\varepsilon}\{\beta\in C : \|\beta-\beta_n\| < \delta\}$$

$$\leq P^{\varepsilon}\{\beta\in C_0(0,1) : \|\beta-\beta_n\| \geq \delta\} + P^{\varepsilon}\{\beta\in C_0(0,1) : \beta_n \in C^{\delta}\}$$

We now estimate each of the probabilities on the right hand side of 5). Let $l_{\delta} = \inf_{\beta\in C^{\delta}} A[\beta]$ then

Bigger than $\beta_n \in C^{\delta}$.

$$P^{\varepsilon}\{\beta\in C_0(0,1) : \beta_n\in C^{\delta}\} \leq P^{\varepsilon}\{\beta\in C_0(0,1) : A[\beta_n] \geq l_{\delta}\}$$

But $A[\beta_n] = \frac{1}{2}\int_0^1 [\beta_n'(s)]^2 ds = \frac{1}{2}\sum_{j=0}^{n-1} n\left|\beta(\frac{j+1}{n}) - \beta(\frac{j}{n})\right|^2$. Therefor

$$P^{\varepsilon}\{\beta\in C_0(0,1) : A[\beta_n] \geq l_{\delta}\} = P\left\{\beta\in C_0(0,1) : \frac{1}{2}\sum_{j=0}^{n-1} n\left|\beta(\tfrac{j+1}{n}) - \beta(\tfrac{j}{n})\right|^2 \geq \frac{l_{\delta}}{\varepsilon}\right\}$$

But $(\beta(\frac{j+1}{n}) - \beta(\frac{j}{n}))\sqrt{n}$ is $N(0,1)$ $= P\{Y_1^2 + \cdots + Y_n^2 \geq \frac{2 l_{\delta}}{\varepsilon}\}$

$Y_1, Y_2, \dots, Y_n$ are indep. $N(0,1)$.

So $Y_1^2+Y_2^2+\cdots+Y_n^2$ is $\chi^2(n)$. So

$$P^\varepsilon\{\beta\in C_0[0,1] : A[\beta_n]\ge l_\delta\} = \frac{1}{\Gamma(\frac{n}{2})2^{n/2}}\int_{\frac{2l_\delta}{\sqrt{\varepsilon}}}^{\infty} x^{\frac{n}{2}-1}e^{-\frac{x}{2}}dx$$

Therefore $P^\varepsilon(C)\le \frac{1}{\Gamma(\frac{n}{2})}\int_{\frac{l_\delta}{\varepsilon}}^{\infty} t^{\frac{n}{2}-1}e^{-t}\,dt + P^\varepsilon\{\beta\in C_0(0,1) : \|\beta-\beta_n\|\ge\delta\}$.

Now $\frac{1}{\Gamma(\frac{n}{2})}\int_{\frac{l_\delta}{\varepsilon}}^{\infty} t^{\frac{n}{2}-1}e^{-t}\,dt \underset{\varepsilon\to 0}{\sim} \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{l_\delta}{\varepsilon}\right)^{\frac{n}{2}-1}e^{-\frac{l_\delta}{\varepsilon}}$ we use the following

Due to Legendre

$$\frac{1}{\Gamma(\frac{n}{2})}\int_{\alpha}^{\infty} t^{\frac{n}{2}-1}e^{-t}\,dt \sim \frac{1}{\Gamma(\frac{n}{2})}\alpha^{\frac{n}{2}-1}e^{-\alpha} \quad \text{as } \alpha\to\infty.$$

Now $P^\varepsilon\{\beta\in C_0(0,1) : \|\beta-\beta_n\|\ge\delta\} \le \sum_{j=1}^{n} P^\varepsilon\{\beta\in C_0(0,1) : \sup_{\frac{j-1}{n}\le s\le\frac{j}{n}}|\beta(s)-\beta_n(s)|\ge\delta\}$

$$\le \sum_{j=1}^{n} P^\varepsilon\{\beta\in C_0(0,1) : \sup_{\frac{j-1}{n}\le s\le\frac{j}{n}}|\beta(s)-\beta(\tfrac{j-1}{n})|\ge\tfrac{\delta}{2}\}$$

(Baroque geometry)

$$= \sum_{j=1}^{n} P\{\beta\in C_0(0,1) : \sup_{\frac{j-1}{n}\le s\le\frac{j}{n}}|\beta(s)-\beta(\tfrac{j-1}{n})|\ge\tfrac{\delta}{2\sqrt{\varepsilon}}\}$$

to catch the extras needed

$$= nP\{\beta\in C_0(0,1) : \sup_{0\le s\le\frac{1}{n}}|\beta(s)|\ge\tfrac{\delta}{2\sqrt{\varepsilon}}\} \le 2nP\{\beta\in C_0(0,1) : \sup_{0\le s\le\frac{1}{n}}\beta(s)\ge\tfrac{\delta}{2\sqrt{\varepsilon}}\}$$

$$= 2n\sqrt{\frac{2}{\pi}}\int_{\frac{\delta\sqrt{n}}{2\sqrt{\varepsilon}}}^{\infty} e^{-\frac{u^2}{2}}du \le 2n\sqrt{\frac{2}{\pi}}\,\frac{2\sqrt{\varepsilon}}{\delta\sqrt{n}}\,e^{-n\delta^2/8\varepsilon}$$, finally

$\lim_{\varepsilon\to 0}\varepsilon\log P^\varepsilon(C)\le \max\left(-l_\delta, -\frac{n\delta^2}{8}\right)$ for any n this holds

hold δ fixed and let $n\to\infty$

$$\le -l_\delta = -\inf_{w\in C^\delta} A[w].$$

Lemma: A is lower semi continuous on $C_0(0,1)$.

Proof: Let $\{\beta_n\}$ be a sequence in $C_0(0,1)$ which converges to $\beta \in C_0(0,1)$ suppose $A[\beta_n] \leq \alpha < \infty$ $\forall n$, we must show $A[\beta] \leq \alpha$. This implies lower semi-continuity. It must be that $\beta_n \in C_0^*(0,1)$ since $A[\beta_n] < \infty$ for each n, we show now that β, the uniform limit of β_n is absolutely continuous.

Let $\{(X_i', X_i'')\}_{i=1}^m$ be disjoint intervals on $[0,1]$. Then

$$\sum_{i=1}^m |\beta_n(X_i'') - \beta_n(X_i')| = \sum_{i=1}^m \left| \int_{X_i'}^{X_i''} \beta_n'(s)\,ds \right|$$

Schwarz inequality $\leq \sum_{i=1}^m \sqrt{\int_{X_i'}^{X_i''} [\beta_n'(s)]^2 ds}\ \sqrt{(X_i'' - X_i')}$

again Schwarz

the whole interval $\leq \sqrt{\sum_{i=1}^m \int_{X_i'}^{X_i''} [\beta_n'(s)]^2 ds}\ \sqrt{\sum_{i=1}^m (X_i'' - X_i')}$

$\leq \sqrt{2\alpha}\ \sqrt{\sum_{i=1}^m (X_i'' - X_i')}$. Let $n \to \infty$ we see

$\sum_{i=1}^m |\beta(X_i'') - \beta(X_i')| \leq \sqrt{2\alpha}\ \sqrt{\sum_{i=1}^m (X_i'' - X_i')}$. Therefore β is absolutely continuous and β' exists a.e. on $[0,1]$. Let $h > 0$ and for each $s \in [0, 1-h]$ we have

$$\left| \frac{\beta_n(s+h) - \beta_n(s)}{h} \right|^2 = \left| \frac{1}{h} \int_s^{s+h} \beta_n'(x)\,dx \right|^2 \quad \text{by Schwarz}$$

$$\leq \frac{1}{h} \int_s^{s+h} [\beta_n'(\tau)]^2 d\tau \quad \text{for each } s.$$

$$\int_0^{1-h} \left| \frac{\beta_n(s+h) - \beta_n(s)}{h} \right|^2 ds \leq \frac{1}{h} \int_0^{1-h} ds \int_s^{s+h} [\beta_n'(\tau)]^2 d\tau$$

$$= \frac{1}{h} \int_0^h d\tau \int_0^{1-h} [\beta_n'(\tau+s)]^2 ds \leq 2\alpha \quad \text{true for}$$

each n we take $n \to \infty$ and uniform convergence implies for each $h > 0$,

$$\int_E f \leq \underline{\lim} \int_E f_n$$

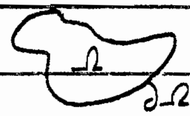
$$\int_0^{1-h} \left| \frac{\beta(s+h) - \beta(s)}{h} \right|^2 ds \leq 2\alpha$$ now we Fatou's lemma

$$\int_0^1 [\beta'(s)]^2 ds \leq 2\alpha$$, taking $\underline{\lim}_{h \to 0}$ inside.

lsing the lower semi-continuity of A, the Arzela-Ascoli theorem
ind the fact that C is closed one can show

$$\lim_{\delta \to 0} \inf_{w \in C^\delta} A[w] = \inf_{w \in C} A[w].$$ So we can

rove the part 1) and we have the action asymptotics.
Recall some classical analysis:



in $\mathbb{R}^2$ $\frac{1}{2}\Delta u + \lambda u = 0$

$u = 0$ on $\partial\Omega$

Jne can prove that there is a discrete set of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots$
(normalized) complete set of eigenvectors $u_1, u_2, u_3, \dots$

$\sum_{\lambda_j < \lambda} 1$ = # eigenvalues $< \lambda$. When $\lambda \to \infty$, Weyl proved

$$\sim \frac{|\Omega|}{2\pi} \lambda \qquad |\Omega| = \int_\Omega dy \int dx$$

lnother famous result is $\sum_{\lambda_j < \lambda} u_j^2(x, y) \sim \frac{\lambda}{2\pi}$ as $\lambda \to \infty$; $(x, y) \in \Omega$.

auberian Theorem: $\int_0^\infty e^{-\lambda t} d\alpha(\lambda) = f(t)$ where
–) is non-decreasing on [, ∞) and the integral exists for $t > 0$.
f we know how $f(t)$ behaves for $t \to 0$ we know how $\alpha(\lambda)$ as $\lambda \to \infty$
" " " $f(t)$ " " $t \to \infty$ " " " $\alpha(\lambda)$ " $\lambda \to 0$.

2/1

Prob.

(Turkish)

Theorem: (Karamata): If for some non negative γ suppose

$$f(t) \sim \frac{c}{t^\gamma} \quad \text{as } t \to 0^+ \quad \text{or as } t \to \infty.$$

Then $\alpha(\lambda) \sim \frac{c\lambda^\gamma}{\Gamma(\gamma+1)}$ as $\lambda \to \infty$ or as $\lambda \to 0^+$.

Try it out on a trivial case: Let n be a positive integer $\alpha(\lambda) = \int_0^\lambda x^n\,dx = \frac{\lambda^{n+1}}{n+1}$

Then $f(t) = \int_0^\infty e^{-\lambda t} \lambda^n \, d\lambda \qquad \lambda t = u$

$$= \frac{1}{t^{n+1}} \int_0^\infty e^{-u} u^n \, du = \frac{\Gamma(n+1)}{t^{n+1}}.$$ Clearly $f(t) \sim \frac{\Gamma(n+1)}{t^{n+1}}$ as $t \to 0$

Then $\gamma = n+1 \quad c = \Gamma(n+1)$ so $\alpha(\lambda) \sim \frac{\Gamma(n+1)\,\lambda^{n+1}}{\Gamma(n+2)} = \frac{\lambda^{n+1}}{n+1}$. This exact. Whoopee doopee we have it.

KAC, Can you hear the shape of a drum?

Prob.

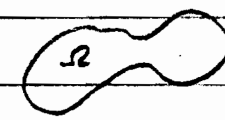
Recall the Tauberian theorem (Karamata): Widder Laplace Transform

$$f(t) = \int_0^\infty e^{-\lambda t}\, d\alpha(\lambda) \quad \text{exists for } t>0$$

and if $\exists\ \gamma \geq 0 \ni f(t) \sim \frac{A}{t^\gamma}$ as $\begin{cases} t \to 0^+ \\ t \to \infty \end{cases}$ and $\alpha(\lambda)$ is a non-decreasing function

Then $\alpha(\lambda) \sim \frac{A\lambda^\gamma}{\Gamma(\gamma+1)}$ as $\begin{cases} \lambda \to \infty \\ \lambda \to 0^+ \end{cases}$ (in contrast with Abelian theorem treats the inverse)

In $\mathbb{R}^2$ we have $\Omega \subset \mathbb{R}$ $\quad \frac{1}{2}\Delta u + \lambda u = 0$



$u = 0$ on $\partial\Omega$

With discrete eigenvalues $\lambda_1, \lambda_2, \cdots$
and normalized eigenfunctions $u_1(x,y), u_2(x,y), \cdots$

$$\text{as } \lambda \to \infty: \sum_{\lambda_j < \lambda} 1 \sim \frac{\lambda}{2\pi}|\Omega| \qquad \text{Weyl}$$

and Carleman proved:

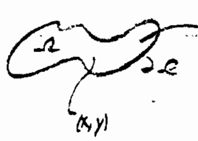
$$\sum_{\lambda_j < \lambda} u_j^2(x,y) \sim \frac{1}{2\pi}\lambda \quad \text{as } \lambda \to \infty.$$

Where does this come from: Vibrating membrane

$F(x,y,t)$ satisfies $\frac{\partial^2 F}{\partial t^2} = \frac{1}{2}\Delta F$ in appropriate ...
we seek $F(x,y,t) = u(x,y)e^{i\omega t}$, then u satisfies

[illegible] $\frac{1}{2}\Delta u + \omega^2 u = 0$ with $u = 0$ on $\partial\Omega$.

Let



start a Brownian motion [illegible] (x_0,y_0) and going hopefully to x,y and set

2/8 ②

Prob.

$p^*(x_0,y_0,x,y,t)$ = the probability that a B.M path in $\mathbb{R}^2$ starting from (x_0,y_0) to (x,y) in time t without touching $\partial\Omega$.

$p(x_0,y_0,x,y,t)$ = the " " " " " " " " " " " " " " " " " ". (Free Brownian motion)

$$p(x_0,y_0,x,y,t) = \frac{1}{2\pi t} e^{-\frac{(x-x_0)^2-(y-y_0)^2}{2t}}$$

. But our intuition says that p^* satisfies

$$\frac{\partial p^*}{\partial t} = \frac{1}{2}\Delta p^*$$

$p^*(x_0,y_0,x,y,t)$ must be singular as $t\to 0$. Or

$$\lim_{t\to\infty} \iint f(x,y)\, p^*(x_0,y_0,x,y,t)\,dx\,dy = f(x_0,y_0)$$

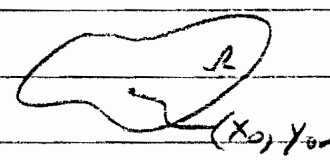
and $p^*(x_0,y_0,x,y,t) = 0$ if $(x,y)\in\partial\Omega$. We can assume a separation argument

$$p^*(x_0,y_0,x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j(x,y)\, u_j(x_0,y_0).$$

This obviously satisfies the above. Now consider

$$p(x_0,y_0,x_0,y_0,t) = \frac{1}{2\pi t}$$ returning in time t.

$$p^*(x_0,y_0,x_0,y_0,t) = \sum_{\lambda=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0,y_0)$$



Obviously for short time

$$\circledast \quad \sum_{\lambda=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0,y_0) \sim \frac{1}{2\pi t} \quad \text{as } t\to 0.$$

Let $\alpha(\lambda) = \sum_{\lambda_j\le\lambda} u_j^2(x_0,y_0)$, then $= \int_0^\infty e^{-\lambda t}\,d\alpha(\lambda) \sim \frac{1}{2\pi t}$ as $t\to\infty$

So by Karamata's theorem, $\gamma=1$ $A=\frac{1}{2\pi}$ we get

2/8 ③

Prob.

$\alpha(\lambda) \sim \frac{1}{2\pi}\lambda$ as $\lambda \to \infty$. Now integrate ⊛ over Ω

we get $\displaystyle\sum_{j=1}^{\infty} e^{-\lambda_j t} \sim \frac{1}{2\pi t}|\Omega|$ as $t \to 0^+$

$$= \int_0^{\infty} e^{-\lambda t}\, d\alpha(\lambda) \qquad \alpha(\lambda) = \sum_{\lambda_j \le \lambda} 1$$

So again by the Tauberian theorem $\alpha(\lambda) \sim \frac{|\Omega|}{2\pi}\lambda$ as $\lambda \to \infty$. We can get new results if we go on. In short time you don't see the boundary, but if you do you see only a piece of straight line. Using the next order of approximation in this way you get

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} \quad \text{as } t \to 0^+$$

Also I. Singer & H. McKean later. On a manifold we have a Laplace-Beltrami operator and with it a diffusion. One can explore the diffusion etc. and get geometrical data.

We proved $P\left\{\overline{\lim_{t \to \infty}} \frac{\beta(t)}{\sqrt{2t\log\log t}} = 1\right\} = 1$ by Khintchine and we

an formulate: For each Brownian motion path $\beta(s)$ $0 \le s < \infty$, $\beta(0) = 0$ define $\beta_t(s) = \frac{\beta(st)}{\sqrt{t\log\log t}}$ for $s \in [0,1]$. So

onsider the family $\{\beta_t(\cdot)$ on $[0,1]$, $(t > 3)\}$. Theorem (Strassen)

or almost all $\beta(\cdot)$, the family β_t, $t > 3$ is compact and

[illegible] then, (as $t \to \infty$).

$$K = \{x(s) \in C_0^1(0,1) \ni \int_0^1 [x'(s)]^2 ds \le 1\}.$$

f this were to be proved we have a Corollary:

Let $\Phi : C_0(0,1) \to \mathbb{R}$ which is continuous, then

$$P\left\{ \overline{\lim_{t\to\infty}}\ \Phi(\beta_t(\cdot)) = \sup_{x\in K} \Phi(x) \right\} = 1, \text{ obviously}$$

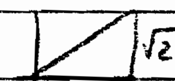
by definition of continuous functional. And

$$P\left\{ \underline{\lim_{t\to\infty}}\ \Phi(\beta_t(\cdot)) = \inf_{x\in K} \Phi(x) \right\} = 1. \text{ Lets look at}$$

some special cases:

1) $\Phi(\omega) = \omega(1)$ then $P\left\{ \overline{\lim_{t\to\infty}}\ \Phi(\beta_t(\cdot)) = \overline{\lim_{t\to\infty}} \frac{\beta(t)}{\sqrt{t\log\log t}} = \sup_{x\in K} x(1) \right\} = 1$

Now the Calculus of variation problem.



so $\Phi(\omega) = \sqrt{2}$ a.e. So this is the law of the iterated logarith.

(2) $I(\omega) = \int_0^1 \omega^2(\tau)\,d\tau$ we get

$$P\left\{ \overline{\lim_{t\to\infty}}\ \frac{1}{t^2\log\log t} \int_0^t \beta^2(s)\,ds = \frac{16}{\pi^2} \right\} = 1.$$

We begin the proof using action asymptotics: K is compact in $C_0(0,1)$ with uniform topology. Choose $g > 1$ and let $n_j = [g^j]$ and consider

(Theorem: Let $\beta(s)$ $0 \le s < \infty$ be B.M. with $\beta(0) = 0$. For positive integers n let

$$\beta_n(s) = \frac{\beta(ns)}{\sqrt{n\log\log n}} \qquad 0 \le s < 1 \quad , n > 3$$

For each $\beta(\cdot) \in C_0(0,1)$ consider the family $\{B_n(\cdot)\}_{n \ge 3}$ on C_0
For almost all $\beta(\cdot)$ the family $\{\beta_n(\cdot)\}_{n=3}^{\infty}$ is compact and the set K of limit points is $K = \{X(s) \in C_0(0,1) : A[X] \le 1\}$.

Prob.

onsider the sequence $\{\beta_{n_j}(\cdot)\}_{j=1,2,\dots}$ for each $\delta > 0$ let K_δ be the sausage of K, i.e. $K_\delta = \bigcup_{x \in K} S(x, \delta)$. Consider

← compliment

$$P\{\beta_{n_j}(\cdot) \in \tilde{K}_\delta\} = P\left\{\frac{\beta(n_j \cdot)}{\sqrt{n_j \log\log n_j}} \in \tilde{K}_\delta\right\}$$

$$= P\left\{\frac{\beta(\cdot)}{\sqrt{\log\log n_j}} \in \tilde{K}_\delta\right\} =$$

$P\{\sqrt{\varepsilon}\,\beta(\cdot) \in \tilde{K}_\delta\}$, $\varepsilon = \frac{1}{\log\log n_j}$ $= P^\varepsilon\{\tilde{K}_\delta\}$, $\tilde{K}_\delta$ is closed

and by action asymptotics $\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log P^\varepsilon\{\tilde{K}_\delta\} \le -\inf_{w \in \tilde{K}_\delta} A[w]$

Therefore for an $r > 0$ $\exists\, \varepsilon > 0$ small enough, i.e. n_j large enough $\ni$

$$P\{\beta_{n_j}(\cdot) \in \tilde{K}_\delta\} = P^\varepsilon(\tilde{K}_\delta) \le e^{-\frac{1}{\varepsilon}\left(\inf_{w \in \tilde{K}_\delta} A[w] - r\right)}$$

Let $\gamma = \inf_{w \in \tilde{K}_\delta} A[w]$. By def$^{\underline{n}}$ of $\tilde{K}_\delta$ we see $\gamma > 1$. Now

hoose $r > 0$ so small that $\gamma^* = \gamma - r > 1$. Thus for sufficiently large j

$$P\{\beta_{n_j}(\cdot) \in \tilde{K}_\delta\} \le e^{-\frac{1}{\varepsilon}\gamma^*} = e^{-\gamma^* \log\log n_j}$$

$$= \frac{1}{(\log n_j)^{\gamma^*}} \quad \text{but}$$

$\frac{1}{(\log n_j)^{\gamma^*}} \sim \frac{\text{const}}{(j \log q)^{\gamma^*}}$ since $\gamma^* > 1$ this is the general term of a convergent series, therefore $\sum_{j=1}^{\infty} P\{\beta_{n_j}(\cdot) \in \tilde{K}_\delta\} < \infty$, since $\gamma^* > 1$. Therefore by the first Borel-Cantelli lemma for almost all β $\exists J \ni \forall j > J \Rightarrow \beta_{n_j}(\cdot) \in K_\delta$. This conclusion holds for any sequence $n_j = [q^j]$ where $q > 1$.

For any integer n let $N(n)$ be the first $n_j \geq n$. Now $\beta_n(s)$ $= \dfrac{\beta(ns)}{\sqrt{n \log\log n}}$ $\beta_N(s) = \dfrac{\beta(Ns)}{\sqrt{N \log\log n}}$ so we onclude $\beta_n(s) = \beta_N(\frac{n}{N}s)\left(\frac{N \log\log N}{n \log\log n}\right)^{1/2}$. Now choose g so close to one that $\frac{n}{N}$ is very close to one. Therefore for almost all β for some n large enough $\beta_n \in K_{2\delta}$, therefore the limit points of $\{\beta_n(\cdot)\}_{n \geq 3}$ are at most the set K.

Probability

onsider $\beta \in C_0(0,1)$ and $\beta_t(s) = \dfrac{\beta(ts)}{\sqrt{t \log\log t}}$ $t \geq 3$ $0 \leq s \leq 1$

or each $\beta \in C_0(0,1)$ we have $\{\beta_t(\cdot)\}_{t \geq 3}$ we proved that every sequence $\{\beta_t(\cdot)\}$ has a convergent subsequence and the set of limit points at most $K = \{x(\cdot) \in C_0(0,1) : A[x] = \frac{1}{2}\int_0^1 [x'(s)]^2 ds \leq 1\}$

onsider the Classical L.I.L.: $P\{\overline{\lim_{t\to\infty}} \frac{\beta(t)}{\sqrt{t \log\log t}} = 1\} = 1$. There are ther iterated logarithm results. What about large deviation in he small direction? We get an action-esque quantity. What is it?

ecall: $E\{e^{-\frac{1}{2}\int_0^t \beta^2(s)ds}\} = \dfrac{1}{\sqrt{\cosh t}}$ so therefore
$\sim \sqrt{2}\, e^{-t/2}$

$$\lim_{t\to\infty} \frac{1}{t} \log E\{e^{-\frac{1}{2}\int_0^t \beta^2(s)ds}\} = -\frac{1}{2}$$ which was explained by (1944)

ac in if $V(x) \geq 0$ and $V(x) \to \infty$ as $|x| \to \infty$ then

$\frac{1}{2}\psi''(x) - V(x)\psi(x) = -\lambda\psi(x)$ has a discrete spectrum

eigenvalues $\lambda_1, \lambda_2, \ldots$ **Thm:** $\lim_{t\to\infty} \frac{1}{t} \log E\{e^{-\int_0^t V(\beta(s))ds}\} = -\lambda_1$
eigenfunctions $\psi_1, \psi_2, \ldots$ (1949)

~~could start at~~ x it is independent of x by ergodicity!

Proof: By Feynman-Kac

$$u(x,t) = E_x\{e^{-\int_0^t V[\beta(s)]ds}\}$$ satisfies

$$u_t = \frac{1}{2}u_{xx} - V(x)u \qquad u(x,0) = 1.$$

On the other hand by separation of variables $u(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\psi_j(x)\int_{-\infty}^{\infty}\psi_j(y)dy$

But they must be equal $E_x\{e^{-\int_0^t V[\beta(s)]ds}\} = \sum_{j=1}^{\infty} e^{-\lambda_j t}\psi_j(x)\int_{-\infty}^{\infty}\psi_j(y)dy$

So $\lim \frac{1}{t} \log E_x\{e^{-\int_0^t V[\beta(s)]ds}\} = -\lambda_1$ since it controls it.

Prob.

ook at $E\{e^{-\int_0^t V[\beta(s)]ds}\}$ as $t\to\infty$,
the paths that stay around zero contribute the most when $V(x)\geq 0$ to the asymptotic behavior of this integral as $t\to\infty$. This is because $V(x)\to\infty$ as $|x|\to\infty$. Other problems in mathematical physics were
$$E\{e^{-\int_0^t\int_0^t \rho(\beta(s),\beta(\tau))\,ds\,d\tau}\},\ \rho(\xi,\eta) \text{ positive definite in } \xi,\eta \text{ as } t\to\infty.$$

What else do we know about λ_1?

$$\lambda_1 = \inf_{\substack{\psi\in L^2\\ \|\psi\|=1}}\left[\int_{-\infty}^{\infty} V(y)\,\psi^2(y)\,dy + \frac{1}{2}\int_{-\infty}^{\infty}[\psi'(y)]^2dy\right].$$ We use the Euler equation,
which is $\frac{1}{2}\psi''(x) - V(x)\psi(x) = -\lambda\psi(x)$!
Since we know this stuff we kind of know:

$$\lim_{t\to\infty}\frac{1}{t}\log E_x\{e^{-\int_0^t V(\beta(s))ds}\} = -\inf_{\psi\in L_2,\ \|\psi\|=1}\left[\int_{-\infty}^{\infty} V(y)\,\psi^2(y)\,dy + \frac{1}{2}\int_{-\infty}^{\infty}[\psi'(y)]^2dy\right]$$

and want to prove it without differential equations. From flat integrals

$$E\{e^{-\int_0^t V(\beta(s))ds}\} \text{ "=" } \int e^{-\int_0^t V(\beta(s))ds}\, e^{-\frac{1}{2}\int_0^t[\beta'(s)]^2ds}\,\delta\beta \quad \text{let } s=t\tau$$

$$= \int e^{-\int_0^1 t\,V(\sqrt{t}\,\beta(\tau))d\tau}\, e^{-\frac{t^2}{2}\int_0^1[\beta'(\tau)]^2d\tau}\,\delta\beta$$

This is not the way. We must be deeper.

Local Time of Brownian Motion: $\beta(s)$ $0\leq s<\infty$; $\beta(0)=x$, B.M. For $t>0$ consider

proportion of time $= \frac{1}{t}\int_0^t \chi_A(\beta(s))ds$ $\quad A\subset\mathbb{R}$, $\chi_A(\ )$ is
up to to t which the path β occupies set A. $\quad$ the indicator function of A.

$$L_t(\beta(\cdot),A) = \frac{1}{t}\int_0^t \chi_A(\beta(s))ds \quad \text{, paths starting at } x$$

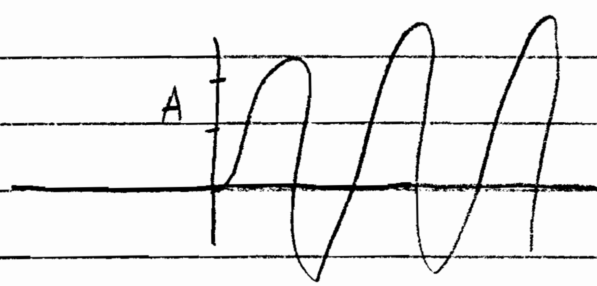
Notice that for fixed $t>0$, fixed $x \in \mathbb{R}$ and particular path $\beta(\cdot)$

$L_t(\beta(\cdot), \cdot)$ is a countably additive, non-negative set function and $L_t(\beta(\cdot), \mathbb{R}) = 1$. So $L_t(\beta(\cdot), \cdot)$ for fixed $x \in \mathbb{R}$, $t>0$ maps $C_x(0,\infty)$ into M the space of probability measures on the real line. Now as a set function, i.e. for fixed $x \in \mathbb{R}$ and $t>0$ for almost all $\beta(\cdot)$ $L_t(\beta(\cdot), \cdot)$ has a density function, call it l_t. We can represent

$$l_t(\beta(\cdot), y) = \frac{1}{t}\int_0^t \delta(\beta(s)-y)\,ds \quad \text{notice}$$

$$L_t(\beta(\cdot), A) = \int_{-\infty}^{\infty} \chi_A(y)\, l_t(\beta(\cdot), y)\,dy$$

$l_t(\beta(\cdot), y)$ is the normalized local time. Look at a picture



$L_t(\beta(\cdot), A) \to 0$ as $t \to \infty$ for any compact set A for almost all $\beta(\cdot)$.

Consider $E_x\left\{e^{-\int_0^t V(\beta(s))\,ds}\right\} = E_x\left\{e^{-t\int_{-\infty}^{\infty} V(y)\, l_t(\beta(\cdot), y)\,dy}\right\}$

whoopee - doopee.

$\int_0^t V(\beta(s))\,ds = t\int_{-\infty}^{\infty} V(y)\,\frac{1}{t}\int_0^t \delta(\beta(s)-y)\,ds\,dy$. For fixed $x \in \mathbb{R}$ and $t>0$

$L_t(\beta(\cdot), \cdot)$ maps $C_x(0,\infty) \to \mathcal{M}$ the space of probability measures on $\mathbb{R}$.

For fixed $x \in \mathbb{R}$ and $t>0$ we now define a probability measure on $\mathcal{M}$ call it $Q_{x,t}$

$$Q_{x,t} = P L_t^{-1} \quad \text{i.e. if } C \subset \mathcal{M}$$

Prob.

$$Q_{x,t}(C) = P\{\beta(\cdot) \in C_x(0,\infty) : L_t(\beta(\cdot),\cdot) \in C\}$$

$L_t(\beta(\cdot),\cdot)$ is the occupation measure. So we can write

$$E_x\left\{e^{-\int_0^t V(\beta(s))ds}\right\} = E_x\left\{e^{-t\int_{-\infty}^{\infty} V(y)\,\ell_t(\beta(\cdot),y)dy}\right\} = E_x\left\{e^{-t\int_{-\infty}^{\infty} V(y)\,dL_t(\beta(\cdot),y)}\right\}$$

$$E_x^{Q_{x,t}}\left\{e^{-t\int_{-\infty}^{\infty} V(y)\,\mu(dy)}\right\} = E_x^{Q_{x,t}}\left\{e^{-t\int_{-\infty}^{\infty} V(y)\,f(y)\,dy}\right\}$$ $\mathcal{F}$ is the space of probability densities on $\mathbb{R}$. We integrate on $\mathcal{F}$ here.

We can now show; by looking at this as $t\to\infty$. We want to see what $Q_{x,t}$ becomes as $t\to\infty$.

To see how $E_x^{Q_{x,t}}\left\{e^{-t\int_{-\infty}^{\infty} V(y)\,f(y)\,dy}\right\}$ behaves as $t\to\infty$ we must see how $Q_{x,t}$ behaves as $t\to\infty$. First of all, how does $L_t(\beta(\cdot),\cdot)$ as $t\to\infty$, answer:

$L_t(\beta(\cdot),A)\to 0$ for $A\subset\mathbb{R}$ compact for a.a. $\beta(\cdot)$.

so for almost all $\beta(\cdot)$ $\ell_t(\beta(\cdot),A)\to 0$ as $t\to\infty$. This is a special case of the ergodic theorem fore one dimensional Brownian Motion. (If this were not B.M. with an invariant measure this for a.e. β converges to the invariant measure)

So $Q_{x,t}$ measure does what? $Q_{x,t}(C)\to 0$ as $t\to\infty$ if $C\neq\mathcal{M}$ ie for reasonable set C.

How fast? **Theorem:** (1) (on $\mathcal{F}$ we need to put Levy' topology)

If $C\in\mathcal{F}$ is closed

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\log Q_{x,t}(C) \le \inf_{f\in C} I(f)$$

If $G\in\mathcal{F}$ is open

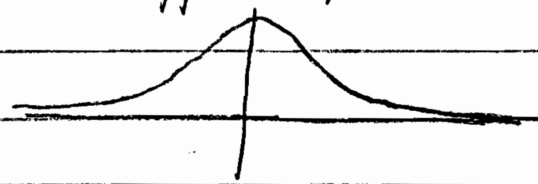
$$\underline{\lim_{t\to\infty}}\ \frac{1}{t}\log Q_{x,t}(G) \ge \inf_{f\in G} I(f) \quad \text{where}$$

2/15

Prob. $Q_{x,t}(C) \sim e^{-\text{const}\, t}$

$\text{const} = \inf_{C \in f} I(f)$

$I(f) = \frac{1}{8} \int_{-\infty}^{\infty} \{ [f'(y)]^2 / f(y) \} dy$. Suppose $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}$



then $I(f) = \frac{1}{8\sigma^2}$.

As $\sigma \to 0$ $I(f) \to \infty$

$\sigma \to \infty$ $I(f) \to 0$. So I tells you how much information is in f, it is called the entropy.

$I : \mathcal{F} \to [0,\infty]$ as a functional.

Roughly speaking as $t \to \infty$ "$Q_{x,t}(A) \sim_{t} e^{-t \inf_{f \in A} I(f)}$" for nice A.

With a particular density function $f \ni A$ and let A shrink on A, so at the point f:

"$Q_{x,t}(f) \sim e^{-t I(f)}$" roughly again.

$$E_x \{ e^{-\int_0^t V(\beta(s)) ds} \} = E^{Q_{x,t}} \{ e^{-t\int_{-\infty}^{\infty} V(y) f(y) dy} \}$$

t large "=" $\int e^{-t\int_{-\infty}^{\infty} V(y) f(y) dy} e^{-t I(f)} \delta f$

$= \int e^{-t [\int_{-\infty}^{\infty} V(y) f(y) dy + I(f)]} \delta f$ as $t \to \infty$ we use Laplace asymptotics:

$$\lim_{t \to \infty} \frac{1}{t} \log E_x \{ e^{-\int_0^t V(\beta(s)) ds} \} = -\inf_{f \in y} [\int_{-\infty}^{\infty} V(y) f(y) dy + \frac{1}{8} \int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)} dy]$$

let $\sqrt{f(y)} = \psi(y)$ then $\int_{-\infty}^{\infty} \psi^2(y) dy = \int_{-\infty}^{\infty} f(y) dy = 1$

So $\psi \in L$ and $\|\psi\| = 1$

$\psi' = \frac{1}{2} f' f^{-1/2}$

$$= -\inf_{\psi \in L^2, \|\psi\|=1} [\int_{-\infty}^{\infty} V(y) \psi^2(y) dy + \frac{1}{2} \int_{-\infty}^{\infty} [\psi'(y)]^2 dy]. \quad [\psi']^2 = \frac{1}{4}\left(\frac{f'^2}{f}\right).$$

Prob.

Using the general structure theorem: Theorem.
Let $\Phi : \mathcal{F} \to \mathbb{R}$ which is bounded and continuous. Then by theorem ① we have

$$\lim_{t\to\infty} \frac{1}{t} \log E^{Q_{x,t}}\left\{ e^{-t\Phi(f)} \right\} = -\inf_{f\in\mathcal{F}} \left[\Phi(f) + I(f) \right] \text{ or}$$

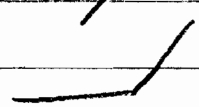
equivalently $$\lim_{t\to\infty} \frac{1}{t}\log E_x\left\{ e^{-t\Phi(l_t(\beta(\cdot),\cdot))} \right\} = \quad '' \quad .$$

But in theorem ① we can only prove that with C is compact and Φ must have special hypotheses. This is more subtle than action asymptotics.

$$\lim_{t\to\infty} \frac{1}{t}\log E^{Q_{x,t}}\left\{ e^{+t\Phi(f)} \right\} = \sup_{f\in\mathcal{F}} \left[\Phi(f) - I(f) \right]$$

↑ always a fight here.

In statistical mechanics thi~~s is α~~ $\Phi(f)$ $= g(\alpha)$ is a convex function of α. There may be an α where there is a phase transition.



α_0

This is due to a non-uniqueness to the f that maximizes this. How do you check uniqueness?

Let $\beta(\cdot)$ be B.M. on $[0,\infty)$ $\beta(0)=0$. For $t \geq 3$ we defined

$$\beta_t(s) = \frac{\beta(st)}{\sqrt{t \log\log t}} \qquad 0 \leq s \leq 1$$

For each $\beta \in C_0(0,\infty)$ we have a family $\{\beta_t(\cdot)\}_{t \geq 3}$. We showed that for almost all $\beta(\cdot) \in C_0(0,\infty)$ every sequence in the family has a convergent subsequence and the set of limit points is at most

$$K = \{x(\cdot) \in C_0(0,1) : A[x] = \tfrac{1}{2}\int_0^1 [x'(s)]^2 ds \leq 1\}$$

We now prove that every $x \in K$ is a limit point of some sequence in $\{\beta_t(\cdot)\}_{t \geq 3}$ for almost all $\beta(\cdot) \in C_0(0,\infty)$.

From the first part, ie that the set of limit points is at most K, we infer that if $\Phi : C_0(0,1) \to \mathbb{R}$ which is continuous then

$$P\left\{ \overline{\lim_{n\to\infty}} \Phi(\beta_n(\cdot)) \leq \sup_{x \in K} \Phi(x) \right\} = 1.$$

For purposes latter consider $\sup\limits_{x \in K} \sup\limits_{0 \leq s \leq T} |x(s)| \leq c\sqrt{T}$ ⊛ $0 \leq T \leq 1$

Let $x(\cdot)$ be any element of K we must exhibit a subsequence of $\{\beta_n(\cdot)\} \ni$ it converges to $x(\cdot)$. Let $\eta > 0$ be given. Choose $k \in \mathbb{Z}$ so large and an $x_0 \in C_0(0,1)$ which is 0 for $0 \leq s \leq \frac{1}{k}$ and

$\frac{1}{2}\int_0^1 [x_0'(s)]^2 ds \leq 1 \ni x_0 \in S(x, \frac{\eta}{2})$. In fact choose k so large that $\frac{c}{\sqrt{k}} < \frac{\eta}{2}$ where c is the constant of the lemma ⊛ Now choose δ so small so that $\frac{c}{\sqrt{k}} + \delta < \frac{\eta}{2}$. We show now that for almost all $\beta(\cdot) \in C_0(0,\infty)$ $\exists$ infinitely many $n \ni$

$$\beta_{n_k}(\cdot) \in S(x_0, \tfrac{c}{\sqrt{k}} + \delta).$$ This will mean that for almost

all $\beta(\cdot) \in C_0(0,\infty)$ $\exists$ infinitely many $n \ni$

Prob.

$\beta_{k^n}(\cdot) \in S(x, \eta)$. For each $\beta(\cdot) \in C_0(0,\infty)$ let

$$\xi_n(s) = \begin{cases} \dfrac{\beta(k^n s) - \beta(k^{n-1})}{\sqrt{k^n \log\log k^n}} & \frac{1}{k} \le s \le 1 \\ 0 & 0 \le s \le \frac{1}{k} \end{cases}$$

Now $P\{\xi_n(\cdot) \in S(x_0, \delta)\} = P\left\{ \sup_{\frac{1}{k} \le s \le 1} \left| \dfrac{\beta(k^n s) - \beta(k^{n-1})}{\sqrt{k^n \log\log k^n}} - x_0(s) \right| < \delta \right\}$

$$= P\left\{ \sup_{\frac{1}{k} \le s \le 1} \left| \frac{\beta(s) - \beta(\frac{1}{k})}{\sqrt{\log\log k^n}} - x_0(s) \right| < \delta \right\}$$ by Brownian Scalining

$$= P^{\varepsilon}\left\{ \sup_{\frac{1}{k} \le s \le 1} \left| \overbrace{\beta(s) - \beta(\tfrac{1}{k})}^{\text{is zero at } s = \frac{1}{k}} - x_0(s) \right| < \delta \right\}$$ $\varepsilon = \dfrac{1}{\log\log k^n}$ as we need.

~~this is an open set~~

From action asymptotics

$$\varliminf_{\varepsilon \to \infty} \varepsilon \log P^{\varepsilon}\left\{ \sup_{\frac{1}{k} \le s \le 1} |\beta(s) - \beta(\tfrac{1}{k}) - x_0(s)| < \delta \right\}$$

$$\ge -A[x_0] \quad \left(\text{actually} \ge -\inf_{x \in} A[x]\right)$$ this set

Choose $\rho > 0$ so small that $\lambda = A[x_0] + \rho < 1$. Therefore

$$P\{\xi_n(\cdot) \in S(x_0, \delta)\} \ge e^{-\frac{1}{\varepsilon}\lambda} \quad \text{for } n \text{ large enough.}$$

$$= e^{-\lambda \log\log k^n} = \frac{1}{(n \log k)^{\lambda}}$$

but $\lambda < 1$ so this is (in n) the general term of a divergent series so

$$\sum_{n=1}^{\infty} P\{\xi_n(\cdot) \in S(x_0, \delta)\} = \infty$$ So by independen

these are independent by incremen

and the Second Borel-Cantelli lemma we have that for

$\xi_n(\cdot) \in S(x_0, \delta)$. Now consider the uniform norm of

$$\|\beta_{k^n}(\cdot) - X_0(\cdot)\| \le \|\beta_{k^n}(\cdot) - \xi_n(\cdot)\| + \|\xi_n(\cdot) - X_0(\cdot)\|.$$

(What is $\beta_{k^n}(s) = \dfrac{\beta(k^n s)}{\sqrt{k^n \log\log k^n}}$ $0 \le s \le 1$?) $\left(\xi_n(\cdot) = \begin{cases} \dfrac{\beta(k^n s) - \beta(k^{n-1})}{\sqrt{k^n \log\log k^n}} & \frac{1}{k} \le s \le 1 \\ 0 & 0 \le s \le \frac{1}{k} \end{cases} \right)$

$$\le \sup_{0 \le s \le \frac{1}{k}} \left| \frac{\beta(k^n s)}{\sqrt{k^n \log\log k^n}} \right| + \|\xi_n(\cdot) - X_0(\cdot)\|.$$

From part one of the theorem for $0 \le T \le 1$

$$\left(\Phi(x(\cdot)) = \sup_{0 \le s \le T} |x(s)| \right)$$

$$\overline{\lim_{n\to\infty}} \sup_{0 \le s \le T} |\beta_n(s)| \le \sup_{x \in K} \sup_{0 \le s \le T} |x(s)|$$

let $T = \frac{1}{k}$ and use the lemma, therefore with Prob. $=1$

$$\overline{\lim_{n\to\infty}} \sup_{0 \le s \le \frac{1}{k}} |\beta_n(s)| \le \frac{c}{\sqrt{k}}$$, therefore for almost all

$\beta(\cdot) \in C_0(0,\infty)$ $\exists$ infinitely many $n \ni \|\beta_{k^n}(\cdot) - X_0(\cdot)\| \le \frac{c}{\sqrt{k}} + \delta$

$< \frac{\eta}{2}$ recall.

So we have it. Q.E.D. Ha-Cha!

Entropy Asymptotics: (A calculation)

Recall $P\left\{ \sup_{0 \le s \le t} \beta(s) \le \alpha \right\} = \sqrt{\frac{2}{\pi t}} \int_0^\alpha e^{-\frac{u^2}{2t}} du$; $\beta(\cdot) \in C_0(0,\infty)$

Therefore $E\left\{ e^{\sup_{0 \le s \le t} \beta(s)} \right\} = h(t) = \int_0^\infty e^\alpha dP\left\{ \sup_{0 \le s \le t} \beta(s) \le \alpha \right\}$

$$= \int_0^\infty e^\alpha \sqrt{\frac{2}{\pi t}} e^{-\frac{\alpha^2}{2t}} d\alpha = \sqrt{\frac{2}{\pi t}} \int_0^\infty e^{-\frac{1}{2t}(\alpha - t)^2} e^{+\frac{t}{2}} d\alpha$$

let $\frac{\alpha - t}{\sqrt{t}} = u$ $= \sqrt{\frac{2}{\pi}} \int_{-\sqrt{t}}^\infty e^{-\frac{u^2}{2}} du \, e^{\frac{t}{2}}$. So we conclude

$\frac{?}{22}$

④

Prob.

$\lim_{t\to\infty} \frac{1}{t} \log h(t) = \frac{1}{2}$. Now we will derive this from action & entropy asymptotics. The theorems are:

ction: $\lim_{\varepsilon\to 0} \varepsilon \log E^{P^\varepsilon}\left\{e^{\frac{1}{\varepsilon}F[\beta(\cdot)]}\right\} = \sup_{w\in C_0^*(0,1) = \{w\in A.C.,\ w'\in L^2\}} \left[F[w] - A[w]\right]$

$\left(\lim_{\varepsilon\to 0} \varepsilon \log E^{P}\left\{e^{\frac{1}{\varepsilon}F[\sqrt{\varepsilon}\beta(\cdot)]}\right\}\right)$

Brownian Scaling

$h(t) = E\left\{e^{\sup_{0\le s\le t}\beta(s)}\right\} = E\left\{e^{\sup_{0\le s\le 1}\beta(st)}\right\} = E\left\{e^{\sup_{0\le s\le 1}\sqrt{t}\,\beta(s)}\right\}$

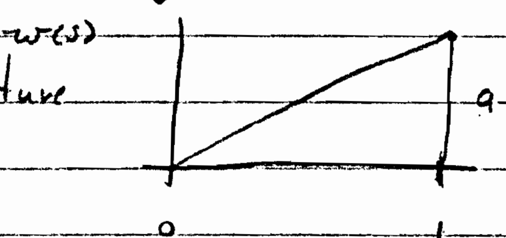
$= E\left\{e^{t \sup_{0\le s\le 1}\frac{1}{\sqrt{t}}\beta(s)}\right\}$ let $\varepsilon = \frac{1}{t}$ then by A.A

$\lim_{t\to\infty} \frac{1}{t}\log h(t) = \lim_{\varepsilon\to 0}\varepsilon \log E\left\{e^{\frac{1}{\varepsilon}\sup_{0\le s\le 1}\sqrt{\varepsilon}\beta(s)}\right\} = \sup_{w\in C_0^*(0,1)}\left[\sup w(s) - \frac{1}{2}\int_0^1 [w'(s)]^2 ds\right]$ (arc length)

We see on the right there is a fight between these two.

$= \sup_{a\ge 0}\left[a - \frac{1}{2}a^2\right] = \frac{1}{2}$ for $a=$

We reduce to this picture



Entropy: Let $\mathcal{F}$ be the space of probability density functions on $\mathbb{R}$, ie $f\ge 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\Phi: \mathcal{F}\to\mathbb{R}$ with hypotheses on Φ. Let

$$l_t(\beta(\cdot), y) = \frac{1}{t}\int_0^t \delta(\beta(s) - y)\,ds$$ is the

[illegible]

$$\lim_{t\to\infty}\frac{1}{t}\log E\left\{e^{t\Phi(l_t(\beta(\cdot),\cdot))}\right\} = \sup_{f\in\mathcal{F}}\left[\Phi[f] - I[f]\right] \text{ when}$$

$$I[f] = \frac{1}{8}\int_{-\infty}^{\infty}\frac{[f'(y)]^2}{f(y)}\,dy.$$

Prob.

For example: if $V(y) \to \infty$ as $|y| \to \infty$ and $\Phi(f) = -\int_{-\infty}^{\infty} V(y) f(y)\,dy$. Then Φ satisfies the unspoken hypotheses. Then

$$\lim_{t\to\infty} \frac{1}{t} \log E\left\{ e^{-t\int_{-\infty}^{\infty} V(y) \frac{1}{t}\int_0^t \delta(\beta(s)-y)\,ds\,dy} \right\} = \sup_{f\in F}\left[-\int_{-\infty}^{\infty} V(y) f(y)\,dy - I[f]\right]$$

$$\lim_{t\to\infty} \frac{1}{t}\log E\left\{ e^{-\int_0^t V(\beta(s))\,ds}\right\} = -\inf_{f\in F}\left[\int_{-\infty}^{\infty} V(y) f(y)\,dy + \frac{1}{8}\int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)}\,dy\right]$$

(Let $\sqrt{f} = \psi$)

or $$\lim_{t\to\infty} \frac{1}{t}\log E\left\{e^{-\int_0^t V(\beta(s))\,ds}\right\} = -\inf_{\substack{\psi\in L^2\\ \|\psi\|=1}}\left[\int_{-\infty}^{\infty} V(y)\psi^2(y)\,dy + \frac{1}{2}\int_{-\infty}^{\infty}[\psi'(y)]^2\,dy\right].$$ But

by Rayleigh-Ritz $= -\lambda_1$ where λ_1 is the lowest e.v. of $\frac{1}{2}\psi'' - V\psi = \lambda\psi$.

Now back to the sup: It was the discovery of Paul Levy that the prob. distribution of $\sup_{0\le s\le t}\beta(s)$ is exactly as the same the p.d.f. of $t\,\ell_t(\beta(\cdot),0)$

ie $$P\left\{\sup_{0\le s\le t}\beta(s) \le \alpha\right\} = P\left\{t\,\ell_t(\beta(\cdot),0)\right\}.$$ We check the first moments at the end of the lecture. Therefore:

$$h(t) = E\left\{e^{\sup_{0\le s\le t}\beta(s)}\right\} = E\left\{e^{t\,\ell_t(\beta(\cdot),0)}\right\} = E\left\{e^{t\,\Phi(\ell_t(\beta(\cdot),\cdot))}\right\}$$

where $\Phi[f] = f(0)$. Therefore from entropy asymptotic

$$\lim_{t\to\infty}\frac{1}{t}\log h(t) = \lim_{t\to\infty}\frac{1}{t}\log E\left\{e^{t\,\ell_t(\beta(\cdot),0)}\right\} = \sup_{f\in F}\left[f(0) - \frac{1}{8}\int_{-\infty}^{\infty}\frac{[f'(y)]^2}{f(y)}\,dy\right]$$

The maximizing family is $f_a(y) = a e^{-2a|y|}$ $a>0$. $\to \sup_{a>0}\left[a - \frac{1}{2}a^2\right] = \frac{1}{2}$

How do you get the maximizing family? Go to $L^2(-\infty,\infty)$

$$\sup\left[f(0) - \int_{-\infty}^{\infty}\frac{[f'(y)]^2}{f(y)}\,dy\right] = \sup\left[\psi^2(0) - \frac{1}{2}\int_{-\infty}^{\infty}[\psi'(y)]^2\,dy\right].$$ Let $\psi(0) = a$

Prob.

So $\Psi''(y) = 2\lambda\Psi$ from the constrained Euler-Lagrange
with $\Psi(0)=a$ problem. So the density that maximizes
it is $e^{-2|y|}$ .

Many equations of mathematical physics can be pinched:

$$\lim_{t\to\infty} \frac{1}{t} \log E\left\{ e^{t\int_{-\infty}^{\infty} \sqrt{\ell_t(\beta(\cdot),y)}\,dy}\right\} = \sup_{f\in\mathcal{F}}\left[\int_{-\infty}^{\infty} \sqrt{f(y)}\,dy - \frac{1}{8}\int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)}\,dy\right].$$

Prob.

Let $C_x[0,\infty)$ be continuous $\beta(s)$ with $\beta(0)=x$, $0\le s<\infty$.
Let P_x and E_x denote B.M. measure and expectation on $C_x[0,\infty)$. Let $t>0$ and $x\in\mathbb{R}$ and define

$$L_t(\beta(\cdot),y) = \frac{1}{t}\int_0^t \chi_{(-\infty,y]}(\beta(s))\,ds$$

note that for fixed x and fixed t * this a p.d.f. in y. * and particular path $\beta(\cdot)$.
Therefore for fixed x and t L_t maps $C_x[0,\infty)\to\mathcal{D}$, the space of prob. dist. functions on $\mathbb{R}$.
Use this mapping to define a probability measure on $\mathcal{D}$.
Let $A\subset\mathcal{D}$, we define the probability measure $Q_{x,t}$

$$Q_{x,t}(A) = P_x\{\beta(\cdot)\in C_x[0,\infty) : L_t(\beta(\cdot),\cdot)\in A\}.$$

this is obviously countably additive in A and $Q_{x,t}(\mathcal{D})=1$.
Recall the topology on $\mathcal{D}$, the Lévy topology, the topology of weak convergence.

<u>Definition</u>: Let $\mathcal{U}$ be the space of functions $u(y)$ on $\mathbb{R}$ which are twice continuously differentiable and for each of which $\exists$ constants α and β $\ni$ $0<\alpha\le u(y)\le\beta<\infty$ $\forall y$.

For all $F\in\mathcal{D}$ define

$$I(F) = -\inf_{u\in\mathcal{U}}\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{u''(y)}{u(y)}\right)dF(y)$$

$\left(\frac{Lu(y)}{u(y)}\right)$ in general

so $I:\mathcal{D}\to[0,\infty]$, also I is lower semi-continuous on $\mathcal{D}$ because of the Lévy topology of weak convergen

$$\int g\,dF_n \to \int g\,dF \qquad \forall g \text{ bounded continuous}$$

We will prove: Let f be a p.d.f. which is continuously differentiable in $\mathbb{R}$ and on the interval $-\infty \le a \le y \le b \le \infty$ f is strictly positive and $f=0$ outside $[a,b]$. For such a density $I(F)$ (where F is the corresponding p.d.f. to F) $= \frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy$ in the sense that if one side is finite then so is the other and they are equal.

Theorem: Let $\Phi : \mathcal{D} \to \mathbb{R}$ satisfying hypotheses I to V, then

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Log} E_x\left\{e^{+t\Phi(L_t(\beta(\cdot),\cdot))}\right\} = \sup_{F\in\mathcal{D}} [\Phi(F) - I(F)]$$

Lemma: Let $C \subset \mathcal{D}$ be compact

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log Q_{x,t}(C) \le -\inf_{F\in C} I(F)$$

[So $Q_{x,t}(C) \sim e^{-t \inf_{F\in C} I(F)}$, as a remark].

Proof: Let $u \in \mathcal{U}$ $\psi(x,t) = E_x\left\{e^{-\int_0^t \frac{1}{2}\frac{u''(\beta(s))}{u(\beta(s))}ds} u(\beta(t))\right\}$, we see Feynman-Kac coming so $\psi(x,t)$ satisfies

$$\psi_t = \frac{1}{2}\psi_{xx} - \frac{1}{2}\frac{u''(x)}{u(x)}\psi, \quad \psi(x,0) = u(x)$$

However the solution is trivially $\psi(x,t) \equiv u(x)$, therefore

$$E_x\left\{e^{-\int_0^t \frac{1}{2}\frac{u''(\beta(s))}{u(\beta(s))}ds} u(\beta(t))\right\} = u(x).$$ Since $u \in \mathcal{U}$, we have

$$E_x\left\{e^{-\int_0^t \frac{1}{2}\frac{u''(\beta(s))}{u(\beta(s))}ds}\right\} \le \frac{u(x)}{\alpha}$$ i.e.

$$E_x\left\{e^{-t\int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}dL_t(\beta(\cdot),y)}\right\} \le \frac{u(x)}{\alpha}$$ because

Prob.

$L_t(\beta(\cdot), y) = \frac{1}{t}\int_0^t \chi_{[-\infty, y]}(\beta(s))\,ds$ with density

$\ell_t(\beta(\cdot), y) = \frac{1}{t}\int_0^t \delta(\beta(s) - y)\,ds$. By the defn. of $Q_{x,t}$ measure

$$E^{Q_{x,t}}\left\{ e^{-t\int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y)} \right\} \le \frac{u(x)}{\alpha} \le \frac{\beta}{\alpha}.$$ This gives us

for any Borel set $C \subset \mathcal{D}$, by a Chebychev-like inequality

$$Q_{x,t}(C) \le \frac{\beta}{\alpha} e^{t \sup_{F \in C} \int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y)},$$ therefore

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log Q_{x,t}(C) \le \sup_{F\in C} \int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y),$$ therefore since

this holds for all $u \in \mathcal{U}$ we can conclude that

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log Q_{x,t}(C) \le \inf_{u\in\mathcal{U}} \sup_{F\in C} \int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y).$$ Suppose

$C \subset \bigcup_{j=1}^{k} C_j$ where each C_j is a Borel set. Then trivially

$$Q_{x,t}(C) \le \sum_{j=1}^{k} Q_{x,t}(C_j)$$ and so

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log Q_{x,t}(C) \le \max_{1\le j\le k} \inf_{u\in\mathcal{U}} \sup_{F\in C_j} \int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y)$$ but this

true for all such coverings by Borel sets of C so

$$(1)\quad \overline{\lim_{t\to\infty}} \frac{1}{t}\log Q_{x,t}(C) \le \inf_{C\subset\bigcup_{j=1}^{k} C_j} \max_{1\le j\le k} \inf_{u\in\mathcal{U}} \sup_{F\in C_j} \int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y).$$

Suppose $C \subset \mathcal{D}$ be compact as in the lemma

We, of course have (1) for this C. Let $l = -\inf_{F\in C} I(F)$, by finition

$$l = \sup_{F\in C}\inf_{u\in\mathcal{U}} \int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y) \quad \therefore \text{ for every } F\in C$$

$$\inf_{u\in\mathcal{U}} \int_{-\infty}^{\infty}\frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y) \le l \quad \therefore \text{ for any } \varepsilon>0 \text{ and } \forall F\in C$$

$\exists\, u_F\in\mathcal{U} \ni \int_{-\infty}^{\infty}\frac{1}{2}\frac{u_F''(y)}{u_F(y)}\,dF(y) \le l+\varepsilon_F$, since $\frac{u''(y)}{u(y)}$ is bounded and continuous $\forall u\in\mathcal{U}$

$\int_{-\infty}^{\infty}\frac{1}{2}\frac{u_F''(y)}{u_F(y)}\,dF(y)$ is a continuous functional on $\mathcal{D}$ (by Lévy topol.) thus for every $F\in C$ $\exists$ a neighborhood $N_F \ni \forall G\in N_F$

$\int_{-\infty}^{\infty}\frac{1}{2}\frac{u_F''(y)}{u_F(y)}\,dG(y) \le l+2\varepsilon$ now the collection N_F for $F\in C$ is an open covering of C, but by compactness $\therefore$ $\exists\, F_1, F_2, \dots, F_k \ni C\subset\bigcup_{j=1}^{k} N_{F_j}$. Let $C_j = N_{F_j}$. $\therefore C\subset\bigcup_{j=1}^{k} C_j$ and we have

$$\sup_{G\in C_j}\int_{-\infty}^{\infty}\frac{1}{2}\frac{u_F''(y)}{u_F(y)}\,dG(y) \le l+2\varepsilon \quad \text{and we have}$$

$$\inf_{u\in\mathcal{U}}\sup_{G\in C_j}\int_{-\infty}^{\infty}\frac{1}{2}\frac{u''(y)}{u(y)}\,dG(y) \le l+2\varepsilon$$ and so also, since it is true for all C_j

$$\max_{1\le j\le k}\inf_{u\in\mathcal{U}}\sup_{G\in C_j}\int_{-\infty}^{\infty}\frac{1}{2}\frac{u''(y)}{u(y)}\,dG(y) \le l+2\varepsilon$$ and so this is true for all such coverings

$$\inf_{C\subset\bigcup_{j=1}^{k}}\max_{1\le j\le k}\inf_{u\in\mathcal{U}}\sup_{G\in C_j}\int_{-\infty}^{\infty}\frac{1}{2}\frac{u''(y)}{u(y)}\,dG(y) \le l+2\varepsilon$$ and by (1) we have it.

Lemma: Let $f(y)$ be a prob. density function on $\mathbb{R}$ which is strictly positive on (a,b) where $-\infty \le a < b \le \infty$ and $f=0$ outside of $[a,b]$ and f is continuously differentiable. Then for F the corresponding dist. f^{n}:

$$I(F) = \frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy$$

in the sense that if the right hand side is finite so is the left and they are equal.

Proof: Assume that $\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy < \infty$. We show now that $\forall u \in \mathcal{U}$

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y) \ge \frac{-1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy$$ which will

imply $I(F) \le \frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy$. For any $u \in \mathcal{U}$ let $h = \log u$ then

$$(*) \quad \int_{-\infty}^{\infty} \frac{1}{2}\frac{u''(y)}{u(y)}\,dF(y) = \frac{1}{2}\int_{-\infty}^{\infty}\left(h''(y) + [h'(y)]^2\right) f(y)\,dy.$$ Now consider

$$\left|\int_a^b h'(y) f'(y)\,dy\right| = \left|\int_a^b \frac{h'(y) f'(y)}{\sqrt{f(y)}}\sqrt{f(y)}\,dy\right|$$ by Schwarz

$$\le 2\left[\int_a^b [h'(y)]^2 f(y)\,dy\right]^{1/2}\left[\frac{1}{4}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy\right]^{1/2}$$

since $2ab \le a^2+b^2$:

$$\le \int_a^b [h'(y)]^2 f(y)\,dy + \frac{1}{4}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy$$ $\therefore$ Rearranging

$$\frac{1}{2}\int_a^b [h'(y)]^2 f(y) - \frac{1}{2}\int_a^b h'(y) f'(y)\,dy \ge -\frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy$$

integration by parts

$$\frac{1}{2}\int_a^b \left\{[h'(y)]^2 + h''(y)\right\} f(y)\,dy \ge -\frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy$$

so we have it by $(*)$.

Now we prove the other way, to show: If $I(F)$ is finite then

$$\frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy \le I(F)$$ which will complete the theorem.

From the definition of $I(F)$ that $\forall u \in \mathcal{U}$ and therefore any $h = \log u$

$$\frac{1}{2}\int_a^b \frac{u''(y)}{u(y)}\,dF(y) = \frac{1}{2}\int_a^b \left\{ [h'(y)]^2 + h''(y)\right\} f(y)\,dy \ge -I(F).$$

In particular this last inequality holds for those h's which are constant outside of a closed interval contained in (a,b). If we let $\mathcal{H}$ denote this latter class of functions h, we have that for any $h \in \mathcal{H}$

$$\frac{1}{2}\int_a^b \left\{ [h'(y)]^2 f(y) - h'(y) f'(y)\right\} dy \ge -I(F).$$

If $h \in \mathcal{H}$ then $\lambda h \in \mathcal{H}$ for any real λ and therefore we have

$$\frac{\lambda^2}{2}\int_a^b [h'(y)]^2 f(y)\,dy - \frac{\lambda}{2}\int_a^b h'(y) f'(y)\,dy + I(F) \ge 0.$$

This cannot have distinct real roots 0 so the discriminant of this quadratic in λ is ≤ 0. This is the $\mathcal{H}$

$$\left|\int_a^b h'(y) f'(y)\,dy\right| \le \sqrt{8I(F)}\left[\int_a^b [h'(y)]^2 f(y)\,dy\right]^{1/2}.$$ Let $g(y) = h'$

Then we have (*) $$\left|\int_a^b g(y) f'(y)\,dy\right| \le \sqrt{8I(F)}\left[\int_a^b g^2(y) f(y)\,dy\right]^{1/2}$$

This is true for all g continuously differentiable and with

3/7

Prob.

ompact support contained in (a,b). But this is also true for any g which s continuous with compact support in (a,b) by Wierstrass approximation.

et $\psi(y) \in C^\infty$ with compact support in (a,b) and $0 \le \phi(y) \le 1$ for all $y \in \mathbb{R}$ We now apply the proceeding inequality to

$$g(y) = \frac{f'(y)\,\psi(y)}{f(y)}$$ which is continuous and has compact support in (a,b).

So $$\left|\int_a^b \frac{[f'(y)]^2}{f(y)} \psi(y)\,dy\right| \le \sqrt{8I(F)}\left[\int_a^b \frac{[f'(y)]^2}{f(y)} \psi(y)\,dy\right]^{1/2}$$ got rid of one $\psi(y)$

we get $$\frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\psi(y)\,dy \le I(F)$$ and $\therefore$ by approximation

of the function 1 by the ψ's we get by bounded convergence

$$\frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy \le I(F).$$ If instead of B.M. if we ave $u_t = Lu$, then

$$I = -\inf_{u \in D^+L} \int \frac{Lu(y)}{u(y)}\,dy$$, if L is a

self-adjoint operator there is an analogous argument that gives you a theorem similar to the lemma.

Prob.

$C_x(0,\infty)$ is cont. $\beta(s)$ $0 \le s < \infty$, $\beta(0) = x$ P_x = B.M. measure, E_x = " " expectation

and $L_t(\beta(\cdot), y) = \frac{1}{t}\int_0^t \chi_{(-\infty, y]}(\beta(s))\,ds$ occupation distribution

For fixed $x \in \mathbb{R}$, $t > 0$ $L_t(\beta(\cdot), \cdot)$ map $C_x(0,t) \to \mathcal{D}$ the space distribution function on $\mathbb{R}$ with the Lévy topology.

Use this mapping to define a measure $Q_{x,t}$.
$A \in \mathcal{D}$ then $Q_{x,t}(A) = P_x\{\beta(\cdot) \in C_x(0,t) : L_t(\beta(\cdot), \cdot) \in A\}$

We proved: If $C \subset \mathcal{D}$ is compact then

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log Q_{x,t}(C) \le -\inf_{F \in C} I(F)$$

where $I(F) : \mathcal{D} \to \mathbb{R}^+ ([0,\infty])$ is defined by

$$I(F) = -\inf_{u \in \mathcal{U}} \frac{1}{2}\int_{-\infty}^{\infty} \frac{u''(y)}{u(y)}\,dF(y)$$ where $\mathcal{U} = \{f(y), -\infty < y < \infty : u \in C^2$ and $\exists$ consts. $\alpha, \beta \ni 0 < \alpha \le u(y) \le \beta < \infty$ $\forall y$ and u'' bounded$\}$

We also proved that

$$I(F) = \frac{1}{8}\int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)}\,dy$$ where f is the corresponding mass $dF(y) = f(y)\,dy$

Theorem: If $\Phi : \mathcal{D} \to \mathbb{R}$ satisfies I-V then

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\{e^{-t\Phi(L_t(\beta(\cdot), \cdot))}\} = -\inf_{F \in \mathcal{D}} [\Phi(F) + I(F)]$$

Lemma: Let $\Phi : \mathcal{D} \to \mathbb{R}$ satisfy hypotheses:

I. $0 \le \Phi(F) \le \infty$
II. Φ is lower semi-continuous on $\mathcal{D}$
III. For any K $\{F \in \mathcal{D} : \Phi(F) \le K\}$ is compact in $\mathcal{D}$.

Example: If $\Phi(F) = \int_{-\infty}^{\infty} V(y)\, dF(y)$ then III forces $V(y) \to \infty$ as $|y| \to \infty$.

Then $\overline{\lim} \frac{1}{t} \log E_x\{e^{-t\Phi(L_t(\beta(\cdot),\cdot))}\} \le -\inf_{F \in \mathcal{D}} [\Phi(F) + I(F)]$.

Proof: For any K let $A_K = \{F \in \mathcal{D} : \Phi(F) \le K\}$ so by III A_K is compact.
Consider

$$E_x\{e^{-t\Phi(L_t(\beta(\cdot),\cdot))}\} \overset{\text{"by defn."}}{=}$$

$$E^{Q_{x,t}}\{e^{-t\Phi(F)}\} = E^{Q_{x,t}}_{A_K}\{e^{-t\Phi(F)}\} + E^{Q_{x,t}}_{\tilde{A}_K}\{e^{-t\Phi(F)}\}$$

$$\le E^{Q_{x,t}}_{A_K}\{e^{-t\Phi(F)}\} + e^{-tK}$$

$$(*)\ \overline{\lim_{t\to\infty}} \tfrac{1}{t} \log E_x\{e^{-t\Phi(L_t(\beta(\cdot),\cdot))}\} \le \max\left(\overline{\lim_{t\to\infty}} \tfrac{1}{t} \log E^{Q_{x,t}}_{A_K}\{e^{-t\Phi(F)}\}, -K\right)$$

We note that by the definition of $I(F)$ it is lower semi-continuous on $\mathcal{D}$ as is $\Phi(F)$ by II. Let $F^* \in A_K$. Assume first $I(F^*) < \infty$, since I is l.s.c. $\exists$ a neighborhood N_{F^*} $\ni$ $\forall F \in N_{F^*}$ and any $\varepsilon > 0$

$$I(F^*) \le I(F) + \varepsilon \quad \text{and also}$$
$$\Phi(F^*) \le \Phi(F) + \varepsilon$$

Let G_{F^*} be another neighborhood of F^* $\ni$ $\overline{G_{F^*}} \subset N_{F^*}$. Consider

3/21 Prob. (3)

inspt

$$E^{Q_{x,t}}_{G_{F^*}\cap A_K}\left\{e^{-t\Phi(F)}\right\} \le e^{-t(\Phi(F^*)-\varepsilon)}\, Q_{x,t}(G_{F^*}\cap A_K)$$

$$\therefore \overline{\lim_{t\to\infty}} \frac{1}{t}\log E^{Q_{x,t}}_{G_{F^*}\cap A_K}\left\{e^{-t\Phi(F)}\right\}$$

$$\le -\Phi(F^*)+\varepsilon - \inf_{F\in \bar{G}_{F^*}\cap A_K} I(F) \le -\Phi(F^*)+\varepsilon - \inf_{F\in N_F^*} I(F)$$

$$\le -\Phi(F^*)+\varepsilon - (I(F^*)-\varepsilon) \le -\inf_{F\in\mathcal{D}}[I(F)+\Phi(F)] + 2\varepsilon$$

On the other hand if $I(F^*)=\infty$ the by l.s.c. of I $\exists$ a nbd. of F^* $M_{F^*} \ni \forall F\in M_{F^*}$ $I(F)\ge K$. Let G_{F^*} be another nbd. of $F^* \ni \bar{G}_{F^*}\subset M_{F^*}$ and we have

$$E^{Q_{x,t}}_{G_{F^*}\cap A_K}\left\{e^{-t\Phi(F)}\right\} \le Q_{x,t}(\bar{G}_{F^*}\cap A_K)$$

$$\therefore \overline{\lim_{t\to\infty}} \frac{1}{t}\log E^{Q_{x,t}}_{G_{F^*}\cap A_K}\left\{e^{-t\Phi(F)}\right\} \le -\inf_{F\in\bar{G}_{F^*}\cap A_K} I(F) \le -K$$

$\therefore$ each point F^* of A_K is covered by a nbd. G_{F^*} (whether $I(F^*)$ is infinite or finite) and since A_K is compact $\exists$ a finite sub-collection

$$F_1^*, F_2^*, \dots, F_n^* \ni A_K = \bigcup_{j=1}^{n} G_{F_j^*} \quad \text{we have that}$$

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log E^{Q_{x,t}}_{A_K}\left\{e^{-t\Phi(F)}\right\} \le \max_{1\le j\le n}\left(\overline{\lim_{t\to\infty}}\frac{1}{t}\log E^{Q_{x,t}}_{G_{F_j^*}\cap A_K}\left\{e^{-t\Phi(F)}\right\}\right)$$

$$\le \max\left(-K, -\inf_{F\in\mathcal{D}}[\Phi(F)+I(F)+2\varepsilon]\right)$$

$$\therefore \text{by } \circledast \quad \overline{\lim_{t\to\infty}}\frac{1}{t}\log E_x\left\{e^{-t\Phi(L_t(\beta(\cdot),\cdot))}\right\} \le \max\left(-K, -\inf_{F\in\mathcal{D}}[\Phi(F)+I(F)+2\varepsilon]\right)$$

So $= -\min\left(K, \inf_{F\in\mathcal{D}} [\Phi(F)+I(F)] - 2\varepsilon\right)$

…t K and ε are arbitrary let $K\to\infty$ as ε is fixed. Now let …$\to 0$ and we have

$$\varlimsup_{t\to\infty} \frac{1}{t}\log E_x\left\{e^{-t\Phi(L_t(\beta(\cdot),\cdot))}\right\} \le -\inf_{F\in\mathcal{D}} [\Phi(F)+I(F)]. \quad \circledast$$

You cannot drop the any of the hypotheses ~~&~~ to the theorem and have …t still true.

…yp. IV: If $F_n \Rightarrow F$ (weak convergence) and the support of F_n is contained …n the finite closed interval $[a,b]$ ~~th~~ then $\Phi(F_n)\to\Phi(F)$.

…et $\mathbb{D}_x$ be the set of all density functions with two continuous …erivatives and for which $\exists$ a closed interval $[a,b]$, $\ni x\in \operatorname{int}[a,b]$ outside …f which $f=0$ and $f>0$ on (a,b) and $\int_a^b \frac{[f'(y)]^2}{f(y)} < \infty$.

…emma: Let Φ satisfy IV. then

$$\varliminf_{t\to\infty} \frac{1}{t}\log E_x\left\{e^{-t\Phi(L_t(\beta(\cdot),\cdot))}\right\} \ge -\inf_{f\in\mathbb{D}_x}\left\{\Phi(f)+\frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy\right\}$$

these depend on a, b.

Under V (to come) this equals $\circledast$. (Ventzel & Freidlin).

Proof: Let $f\in\mathbb{D}_x$ and let F be the corresponding dist. function. For each $\beta(\cdot)\in C_x(0,t)$ $\ni$ $a\le\beta(s)\le b$ $\forall\, 0\le s\le t$, its occupation distribution function whose support is contained in $[a,b]$.
$L_t(\beta(\cdot),y)$

IV for any $\varepsilon > 0$ $\exists$ a nbd of F, $N_F \ni$ if $L_t(\beta(\cdot), y) \in N_F$ any $\beta(\cdot) \ni a \le \beta(s) \le b$ $\forall 0 \le s \le t$ then

$$|\Phi(L_t(\beta(\cdot), \cdot) - I(F)| < \varepsilon$$

Let $C^t_{a,b} = \{\beta(\cdot) \in C_x(0,t) : a \le \beta(s) \le b \ \forall 0 \le s \le t\}$.

Now
$$E_x\{e^{-t\Phi(L_t(\beta(\cdot), \cdot))}\} \ge E_x\{e^{-t\Phi(L_t(\beta(\cdot), \cdot))}; L_t(\beta(\cdot), \cdot) \in N_F \text{ and } \beta(\cdot) \in C^t_{a,b}\}$$

set of integrands.

$$\ge e^{-t(I(F)+\varepsilon)} P_x\{L_t(\beta(\cdot), \cdot) \in N_F; \beta(\cdot) \in C^t_{a,b}\}.$$

We will show that
$$\lim_{t\to\infty} \frac{1}{t} \log P_x\{L_t(\beta(\cdot), \cdot) \in N_F; \beta(\cdot) \in C^t_{a,b}\} \ge -\frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy \quad (**)$$

we will show this so procede young man:

$$\therefore \lim_{t\to\infty} \frac{1}{t} \log E_x\{e^{-t\Phi(L_t(\beta(\cdot), \cdot))}\} \ge -(\Phi(F)+\varepsilon) - \frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy$$

since $\varepsilon > 0$ is arbitrary
$$\ge -\left[\Phi(f) + \frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy\right]$$

But $f \in \mathbb{D}_x$ is arbitrary so
$$\ge -\inf_{f \in \mathbb{D}_x}\left[\Phi(f) + \frac{1}{8}\int_a^b \frac{[f'(y)]^2}{f(y)}\,dy\right]$$

Now the sticky proof: Recall $u_t = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ has f. solution
$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}.$$

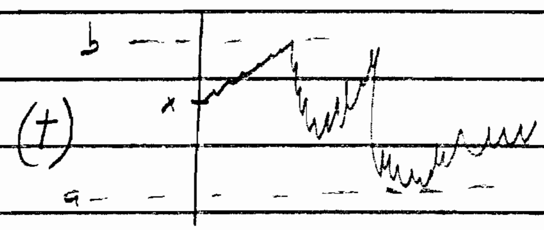
[illegible] u_{xx} [illegible] B.M with drift.
Let $p^*(t,x,y)$ be the thing for the drift on continuous functions. Are they A.C. w.r.t. each other? Yes. And they have Radon-Nikodym derivative given by

Cameron-Martin:

$$E_x\{G[\beta]\} = \tilde{E}_x\{G[\beta]\, e^{\underbrace{-\int_0^t b(\beta(s))\,d\beta(s) + \frac{1}{2}\int_0^t b^2(\beta(s))\,ds}_{\text{not classically defined.}}}\}$$

Radon-Nikodym derivative

We are trying to prove: (**) Let $f \in \mathbb{D}_x$ and let F be the c.d.f. and let N_F be any Lévy nbd of F. Let $b(y) = \frac{1}{2}\frac{f'(y)}{f(y)}$ and consider B.M. with this drift b. So the drift is like



it can't get out. Look at Ito, McKean under Feller's test.

The deepest result of Markov chains is ergodic theory. The invariant measure above is $f(y)$. That is

$$L_t(\beta(\cdot), y) \xrightarrow{\text{weak}} F(y) \quad \text{for a.a. } \beta \text{ in drift measure.}$$

From Cameron-Martin formula

$$P_x\{\beta \in C_x(0,t) : L_t(\beta(\cdot),\cdot) \in N_{F},\ \beta(\cdot) \in C^t_{a,b}\}$$

we can drop this because of (†)

$$= \tilde{E}_x\{e^{-\int_0^t b(\beta(s))\,d\beta(s) + \frac{1}{2}\int_0^t b^2(\beta(s))\,ds};\ L_t(\beta(\cdot),\cdot) \in N_F,\ \beta(\cdot) \in C^t_{a,b}\}$$

$$= \tilde{E}_x\{e^{-\int_0^t b(\beta(s))\,d\beta(s) + \frac{1}{2}\int_0^t b^2(\beta(s))\,ds};\ L_t(\beta(\cdot),\cdot) \in N_F\}$$

$$= \tilde{E}_x\{e^{-\int_0^t \frac{1}{2}\left(\frac{f'}{f}\right)(\beta(s))\,d\beta(s) + \frac{1}{8}\int_0^t \left(\frac{f'}{f}\right)(\beta(s))\,ds};\ L_t(\beta(\cdot),\cdot) \in N_F\}$$

Prob. (7)

to's Formula: Let $g(y)$ be twice continuously differentiabl

$$dg(\beta(s)) = g'(\beta(s))\, d\beta(s) + \frac{1}{2} g''(\beta(s))\, ds.$$

Prob.

Problem 8. Let $\lambda > 0$ and consider

$$E\left\{e^{\lambda\int_0^t \beta(s)ds - \frac{1}{2}\int_0^t \beta^2(s)ds}\right\}$$

Find $\lim_{t\to\infty} \frac{1}{t}\log E\left\{e^{\lambda\int_0^t \beta(s)ds - \frac{1}{2}\int_0^t \beta^2(s)ds}\right\}$ using entropy asymptotics.

9. Check your result in #8 by actually calculating the function space integral.

N_F is a Levy' nbd of $f \in D_x$

Lemma: $\lim_{t\to\infty}\frac{1}{t}\log P_x\{\beta(\cdot)\in C_x(0,t) : L_t(\beta(\cdot),\cdot)\in N_F,\ \beta(\cdot)\in C_{a,b}^t\} \geq -\frac{1}{8}\int \frac{(f'(y))^2}{f(y)}dx$

Recall: $\frac{1}{2}\frac{d^2}{dy^2} + b(y)\frac{d}{dx}$ $\quad b(y) = \frac{1}{2}\frac{f'(y)}{f(y)}$, w.r.t. the drift property the measure is $\tilde{P}_x$ & $\tilde{E}$ and the invariant measure is f itself.

Using the Cameron-Martin formula we got.

$$P_x\{\beta(\cdot)\in C_x(0,t) : L_t(\beta(\cdot),\cdot)\in N_F,\ \beta(\cdot)\in C_{a,b}^t\}$$
$$= \tilde{E}_x\left\{e^{-\frac{1}{2}\int_0^t\left(\frac{f'}{f}\right)(\beta(s))d\beta(s) + \frac{1}{8}\int_0^t\left(\frac{f'}{f}\right)^2(\beta(s))ds}\ ;\ L_t(\beta(\cdot),\cdot)\in N_F\right\}$$

Ito's Formula: Given any g with two continuous derivatives

$$dg(\beta(s)) = g'(\beta(s))\,d\beta(s) + \frac{1}{2}g''(\beta(s))\,ds$$

Consider $\int_0^t g'(\beta(s))\,d\beta(s)$, in particular let $g(y) = \log f(y)$:

Prob.

$$d \log f(\beta(s)) = (\log f)'(\beta(s))\, d\beta(s) + \tfrac{1}{2} (\log f)''(\beta(s))\, ds$$

integrate from 0 to t with $\beta(0) = x$

$$\log f(\beta(t)) - \log f(x) = \int_0^t \left(\frac{f'}{f}\right)(\beta(s))\, d\beta(s)$$
$$+ \frac{1}{2}\int_0^t \left(\frac{f''}{f}\right)(\beta(s))\, ds - \frac{1}{2}\int_0^t \left(\frac{f'}{f}\right)^2 (\beta(s))\, ds$$

Now we use this to replace the nasty term: $\sigma(x, N_F, t, a, b) =$

$$P_x\{\beta(\cdot) \in C_x(0,t) : L_t(\beta(\cdot), \cdot) \in N_F,\ \beta(\cdot) \in C_{a,b}^t\}$$
$$= \tilde{E}_x\left\{\sqrt{\frac{f(x)}{f(\beta(t))}}\, e^{-\frac{1}{8}\int_0^t \left(\frac{f'}{f}\right)^2(\beta(s))ds + \frac{1}{4}\int_0^t \left(\frac{f''}{f}\right)(\beta(s))ds};\ L_t(\beta(\cdot), \cdot) \in N_F\right\}$$

Let $J(y) = \frac{1}{8}\left(\frac{f'}{f}\right)^2(y) - \frac{1}{4}\left(\frac{f''}{f}\right)(y)$ and $0 < \varepsilon$ be given.

Let $S(t,\varepsilon) = \left\{\beta(\cdot) \in C_x(0,t) : \left|\int_a^b J(y)\, dL_t(\beta(\cdot), y) - \int_a^b J(y) f(y)\, dy\right| < \varepsilon\right\}$

$\tilde{P}(S(t,\varepsilon)) \to 1$ as $t \to \infty$ because $\underbrace{L_t(\beta(\cdot),y) \to f(y)dy}_{\text{invariant measure.}}$

Let $S'(t,\varepsilon) = S(t,\varepsilon) \cap \{\beta(\cdot) \in C_x(0,t) : L_t(\beta(\cdot), \cdot) \in N_F\}$

Note $\int_a^b J(y)\, dL_t(\beta(\cdot), y) = \int_a^b \left(\frac{1}{t}\int_0^t \delta(\beta(s) - y)\, ds\right) dy$

$$= \frac{1}{t}\int_0^t J(\beta(s))\, ds$$

$$\sigma = \tilde{E}_x\left\{\sqrt{\frac{f(x)}{f(\beta(t))}}\, e^{-t\int_a^b J(y)\, dL_t(\beta(\cdot), y)};\ L_t(\beta(\cdot), \cdot) \in N_F\right\}$$
$$\geq \tilde{E}_x\left\{\sqrt{\frac{f(x)}{f(\beta(t))}}\, e^{-t\int_a^b J(y)\, dL_t(\beta(\cdot), y)};\ S'(t,\varepsilon)\right\}$$

$$\geq e^{-t\{\varepsilon + \int_a^b J(y) f(y)\,dy\}} \tilde{E}_x\left\{\sqrt{\tfrac{f(x)}{f(\beta(t))}} \,;\, S'(t,\varepsilon)\right\}$$

Now $\tilde{E}_x\left\{\sqrt{\tfrac{f(x)}{f(\beta(t))}},\ S'(t,\varepsilon)\right\} \geq \gamma\, \tilde{P}_x(S'(t,\varepsilon))$

lower bound by definition

$\exists\ \gamma \ni$ for almost all $\beta(\cdot)$ ($\tilde{P}$-measure) $\sqrt{\tfrac{f(x)}{f(\beta(t))}} \geq \gamma$. So

$$\sigma \geq e^{-t\varepsilon} e^{-t\int_a^b J(y) f(y)\,dy}\ \gamma\, \tilde{P}_x(S'(t,\varepsilon)).$$

Now let $S^c(t,\varepsilon)$ be the complement of $S(t,\varepsilon)$. Then

$$\lim_{t\to\infty} \tilde{P}_x(S^c(x,t)) = \lim_{t\to\infty} \tilde{P}_x\left\{\beta(\cdot)\in C_x(0,t) : \left|\int_a^b J(y)\, dL_t(\beta(\cdot),\cdot) - \int_a^b J(y) f(y)\,dy\right| \geq \varepsilon\right\}$$

Because the ergodic theorem which says that $L_t \underset{\text{weak conv.}}{\Rightarrow} f$

Moreover $\lim_{t\to\infty} \tilde{P}_x\{\beta(\cdot)\in C_x(0,t) : L_t(\beta(\cdot),\cdot) \notin N_F\} = 0$ by the ergodic theorem since this is a Levy neighborhood.

$$\text{But } S'^{c}(t,\varepsilon) = S^c(t,\varepsilon) \cup \{\beta(\cdot)\in C_x(0,t) : L_t(\beta(\cdot),\cdot) \notin N_F\}$$

$$\therefore \lim_{t\to\infty} \tilde{P}_x\{S'^{c}(t,\varepsilon)\} = 0$$

$$\therefore \lim_{t\to\infty} \frac{1}{t}\log\sigma \geq -\varepsilon - \frac{1}{8}\int_a^b \frac{(f'(y))^2}{f(y)}\,dy$$ let $\varepsilon \to 0$ and we have the desired result. With a mild extra hypothesis one can get the upper and the lower estimate are the same. So what we have proved is

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\left\{ e^{+t\Phi(L_t(\beta(\cdot),\cdot))}\right\} = \sup_{F\in\mathcal{D}} \left[\Phi(F) - I(F)\right]$$

$$I(F) = I(f) = \frac{1}{8}\int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)}\,dy$$

<u>Malliavin Calculus:</u> Frechet, Volterra, Gateaux the derivative in function space.

$G: C_0(0,1) \to \mathbb{R}$ and from the Calculus of Variations

$\gamma \in C_0(0,1)$

$$\delta G_{/\gamma}(\beta) = \frac{d}{dh} G[\beta + h\gamma]\Big|_{h=0}.$$ This is a linear functional of γ

$$= \int^1 \left(\frac{\delta G}{\delta \beta(s)}\right) \gamma(s)\, ds$$ ← a generalization of the partial derivative

Lévy called this "concrete functional analysis", also Daniell worked.

Is there a nice relationship between the derivative and the integral?

$G: C_0(0,1) \to \mathbb{R}$

$$E\left\{\frac{\delta G}{\delta\beta(s)}\right\}$$ on scratch paper

$$E\left\{\frac{\delta G}{\delta\beta(s)}\right\} \text{"="} \int \frac{\delta G}{\delta\beta(s)} e^{-\frac{1}{2}\int_0^1 [\beta'(\tau)]^2 d\tau} \delta\beta$$ "integrate by parts"

$$= -\int G[\beta]\,\beta''(s)\, e^{-\frac{1}{2}\int_0^1 [\beta'(\tau)]^2 d\tau} \delta\beta$$

$$= -\frac{d^2}{ds^2}\int G[\beta]\,\beta(s)\, e^{-\frac{1}{2}\int_0^1 [\beta'(\tau)]^2 d\tau} \delta\beta = -\frac{d^2}{ds^2} E\{\beta(s)\, G[\beta]\}.$$ So

we [illegible]

$$E\left\{\frac{\delta G}{\delta\beta(s)}\right\} = -\frac{d^2}{ds^2} E\{\beta(s)\, G[\beta]\}$$ we also see by using inverse

$$\int^1 \min(\tau, s)\, E\left\{\frac{\delta G}{\delta\beta(s)}\right\} ds = E\{\beta(\tau)\, G[\beta]\}$$

Examples

$G[\beta] = \beta(\sigma) \quad \sigma \in [0,1]$

$$E\{\beta(\tau)\, G[\beta]\} = E\{\beta(\tau)\beta(\sigma)\} = \min(\tau,\sigma)$$

$$\frac{\delta G[\beta]}{\delta\beta(s)} = \delta_\sigma(s) \quad \text{then} \quad \int_0^1 \min(\tau,s)\, E\{\delta_\sigma(s)\}\, ds = \min(\tau,\sigma).$$

The proof is four lines long using the Cameron translation formula.
The n-dimensional Laplacian is

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial\beta_j^2} \quad \text{"going all the way"} \quad \int_0^1 \frac{\delta^2}{\delta\beta(s)^2}\, ds \quad \text{but this is toying.}$$

Now we work a little

$$E\{\beta^2(\tau)\, G[\beta]\} = \int_0^1 \min(\tau,s)\, E\left\{\beta(\tau) \frac{\delta G}{\delta\beta(s)}\right\}$$

$$+ \int_0^1 \min(\tau,s)\, E\{G[\beta]\}\, \delta_s(\sigma)\, ds \qquad \text{repeating}$$

$$= \int_0^1\int_0^1 \min(\tau,s_1)\min(\tau,s_2)\, E\left\{\frac{\delta^2 G}{\delta\beta(s_1)\,\delta\beta(s_2)}\right\} ds_1\, ds_2 + \tau E\{G[\beta]\}$$

Consider $E\{h(\beta(\tau))\, G[\beta]\} = E\left\{\frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} h(y)\, e^{-\frac{1}{2\tau}\left(y - \underbrace{\int_0^1 \min(\tau,s) \frac{\delta}{\delta\beta(s)}\, ds}_{\text{this is an operator}}\right)^2} G[\beta]\right\}$

If we put in an exponential we get a nice formula. Let

$h(y) = e^{\mu y}$ moment generating function

$$E\{e^{\mu\beta(\tau)} G[\beta]\} = E\left\{\left(e^{\mu\int_0^1 \min(\tau,s) \frac{\delta}{\delta\beta(s)}}\right) G[\beta]\right\} e^{\frac{\mu^2\tau}{2}}.$$

Why do we want such a thing? Let $g \in C_0(0,1)$ and

$$u(q) = E\left\{ e^{i\int_0^1 q(s)\beta(s)ds - \frac{1}{2}\int_0^1 \beta^2(s)ds} \right\}$$

Let $\tau \in [0,1]$ $\quad \frac{\delta u}{\delta q(\tau)} = i\,E\left\{ \beta(\tau)\, \underbrace{e^{i\int_0^1 q(s)\beta(s)ds - \frac{1}{2}\int_0^1 \beta^2(s)ds}}_{G[\beta]} \right\}$

$$= i\int_0^1 \min(\tau,\sigma)\, E\left\{ [iq(\sigma) - \beta(\sigma)]\, e^{i\int_0^1 q(s)\beta(s)ds - \frac{1}{2}\int_0^1 \beta^2(s)ds} \right\} d\sigma$$

$$= -\left(\int_0^1 \min(\tau,\sigma)\, q(\sigma)\, d\sigma\right) u(q) - \int_0^1 \min(\tau,\sigma)\, \frac{\delta u}{\delta q(\sigma)}\, d\sigma$$

with $u(0) = \frac{1}{\sqrt{\cosh(1)}}$. How do we solve these? This is a Fourier transform in function space. This is a tough thing.

Prob.

I: E w.r.t. B.M. } there is a connection
II: p.d.e. }

Let $\beta(s)$; $0 \le s < \infty$, $\beta(0) = 0$ and $g(s) \in C_0(0,\infty)$ and define

I:

$$u(g;x,t) = E\left\{ \exp\left(i\int_0^t g(s)\beta(s)\,ds - \frac{1}{2}\int_0^t \beta^2(s)\,ds\right)\delta(\beta(t)-x)\right\}$$

we can explicitly get, by random Fourier Series

$$u(g;x,t) = \frac{1}{\sqrt{2\pi\sinh t}}\exp\left(\frac{-x^2}{2\tanh t} - \frac{ix}{\tanh t}\int_0^t R(t,\sigma,-1)g(\sigma)\,d\sigma + \right.$$

$$\left. + \frac{1}{2}\int_0^t\int_0^t R(\sigma,\xi,-1)g(\sigma)g(\xi)\,d\sigma\,d\xi + \frac{1}{2\tanh t}\int_0^t\int_0^t R(t,\sigma,-1)R(t,\xi,-1)g(\sigma)g(\xi)\,d\sigma\,d\xi\right)$$

where $R(\sigma,\xi,-1) = \begin{cases} -\dfrac{\cosh(t-\xi)\sinh\sigma}{\cosh t} & \sigma \le \xi \\ -\dfrac{\cosh(t-\sigma)\sinh\xi}{\cosh t} & \sigma \ge \xi \end{cases}$

II: also $u(g;x,t)$ solves
$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{x^2}{2}u + ixg(t)u$$
$$u(g;x,t) \to 0 \quad \text{as} \quad |x| \to \infty$$
$$u(g;x,t) \to \delta(x) \quad \text{as} \quad t \to 0.$$

<u>Theorem</u>: $u(g;x,t)$ is the unique solution of

$$0 < \tau < t \qquad \frac{\delta u}{\delta g(\tau)} = \left(-\int_0^t \min(\tau,s)g(s)\,ds\right)u$$

$$-\int_0^t \min(\tau,s)\frac{\delta u}{\delta g(s)}\,ds - i\tau\frac{\partial u}{\partial x} \quad \text{with}$$

$$\lim_{\tau\to t^-}\frac{\delta u}{\delta g(\tau)} = ixu$$

$$\frac{\partial u(0;x,t)}{\partial t} - \frac{1}{2}\frac{\partial^2 u(0;x,t)}{\partial x^2} = \frac{-x^2}{2}u(0;x,t) \quad ; \; u(0;x,t) \to \delta(x) \text{ as } t \to 0.$$

4/4

②

Prob.

$$u(q;x,t) = E\left\{ e^{i\int_0^t q(s)\beta(s)\,ds - \frac{1}{2}\int_0^t \beta^2(s)\,ds}\, \delta(\beta(t)-x) \right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu x} E\left\{ e^{i\int_0^t q(\sigma)\beta(\sigma)\,d\sigma - \frac{1}{2}\int_0^t \beta^2(\sigma)\,d\sigma + i\mu\beta(t)} \right\} d\mu$$

Let $\tau \in (0,t)$ & differentiate w.r.t. q:

$$\frac{\delta u}{\delta q(\tau)} = \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu x} E\Big\{ \beta(\tau) \underbrace{e^{i\int_0^t q(\sigma)\beta(\sigma)\,d\sigma - \frac{1}{2}\int_0^t \beta^2(\sigma)\,d\sigma + i\mu\beta(t)}}_{G[\beta]} \Big\} d\mu$$

Recall: $\displaystyle \int_0^t \min(\tau,s)\, E\left\{ \frac{\delta G[\beta]}{\delta \beta(s)} \right\} ds = E\{ \beta(\tau)\, G[\beta] \}$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu x} \int_0^t \min(\tau,s)\, E\left\{ \frac{\delta G[\beta]}{\delta \beta(s)} \right\} ds$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu x} \int_0^t \min(\tau,s)\, E\{ [iq(s) - \beta(s) + i\mu\delta_t(s)]\, G[\beta] \}\, ds\, d\mu$$

So
$$\frac{\delta u}{\delta q(\tau)} = \left(-\int_0^t \min(\tau,s)\, q(s)\,ds \right) u - \int_0^t \min(\tau,s) \frac{\delta u}{\delta q(s)}\, ds - i\tau \frac{\partial u}{\partial x}$$

computing

$$\frac{\delta u}{\delta q(\tau)} = i E\left\{ \beta(\tau) e^{i\int_0^t q(s)\beta(s)\,ds - \frac{1}{2}\int_0^t \beta^2(s)\,ds} (\delta(\beta(t)-x)) \right\}$$

$= i\,u$ ... The rest follows from the Feynman-Kac formula.

That if instead of $\frac{1}{2}\int_0^t \beta^2(s)\,ds$ was replaced by $\int_0^t V(\beta(s))\,ds$

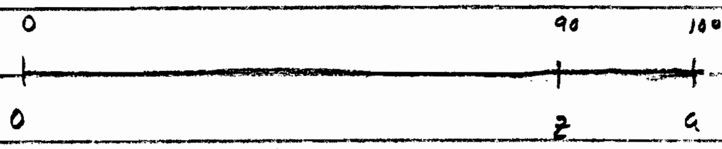
Prob.

uniquely determine $K(t) = \frac{1}{\sqrt{2\pi \sinh t}}$ and so if we iterate this argument we get $c_1, c_2, c_3, \ldots$ etc and we get the whole expression for $u(q; x, t)$.

<u>Markov Chains</u>: The Gambler's ruin problem:

prob. of winning $= p$
" " losing $= q$ $\quad p+q=1$

Starts with z dollars + plays until he has 0 or a dollars



suppose $p = q = \frac{1}{2}$.

et $q(z) =$ probability of his being ruined here $= .9$
" " " succeeding $.1$

$q(z) = p\,q(z+1) + q\,q(z-1)$ where $q(a) = 0$, $q(0) = 1$

are 1: $p = q = \frac{1}{2}$ $\quad q(z) = \frac{1}{2}\left(q(z+1) + q(z-1)\right)$ $\quad q(a) = 0,\ q(0) = 1$

particular solutions $1, z$ so general solution

$q(z) = A + Bz$

$q(z) = 1 - \frac{z}{a}$ which is what it should be intuitively.

se 2: $p < q$ $\quad q(z) = p\,q(z+1) + (1-p)\,q(z-1)$ $\quad q(a) = 0$, $q(0) = 1$

articular solutions 1, and $\left(\frac{q}{p}\right)^z = q(z)$

Does $\left(\frac{q}{p}\right)^z \overset{?}{=} p\left(\frac{q}{p}\right)^{z+1} + q\left(\frac{q}{p}\right)^{z-1}$ so the general solution is:

Adv. Top. Prob.

Recall: Let $\xi_1, \xi_2, \cdots$ be i.r.v. with means 0 and variances σ_i^2

$S_i = \xi_1 + \cdots + \xi_i$, $t_i^2 = \sigma_1^2 + \cdots + \sigma_i^2$, then

$$P\{\max_{1 \le i \le m} |S_i| \ge \lambda t_m\} \le 2P\{|S_m| \ge (\lambda - \sqrt{2})t_m\}$$

note: $\sum_{i=1}^{m} P\{E_i\} = P\{\max_{1 \le j \le m} |S_j| \ge \lambda t_m\}$, it follows.

Cor: Let $\xi_1, \cdots,$ be iid r.v.'s with mean 0 and variance, for $\varepsilon > 0$ $\exists$ λ sufficiently large

$$\overline{\lim_{n\to\infty}} P\{\max_{1 \le i \le n} |S_i| \ge \lambda \sigma \sqrt{n}\} < \frac{\varepsilon}{\lambda^2}.$$

Proof: If $\lambda > 2\sqrt{2}$ then then $\lambda - \sqrt{2} \ge \frac{\lambda}{2}$ and from the lemma with $\lambda > 2\sqrt{2}$

$$P\{\max_{1 \le i \le n} |S_n| \ge \lambda \sigma \sqrt{n}\} \le 2P\{|S_n| \ge \frac{\lambda}{2} \sigma \sqrt{n}\} \quad \text{from C.L.T.}$$

$$P\{\frac{|S_n|}{\sigma\sqrt{n}} \ge \frac{\lambda}{2}\} \to P\{|N| \ge \frac{\lambda}{2}\} \qquad N \text{ is } N(0,1)$$

$$\le 8\left(\frac{1}{\lambda^3}\right) E\{|N|^3\}$$

Now choose λ so large that $\varepsilon > 2 \cdot \frac{8}{\lambda} E\{|N|^3\}$ and we have it.

Heuristics on why this should be true: $F: C[0,1] \to \mathbb{R}$ then

$$P\{F[\beta] \le \lambda\} = \lim_{n\to\infty} \{F[x_n(\cdot)] \le \lambda\} \qquad x_n(\tau) = \begin{cases} \frac{S_1}{\sqrt{n}} & \tau = 0 \\ \frac{S_i}{\sqrt{n}} & \frac{i-1}{n} < \tau \le \frac{i}{n} \quad i \ge 1 \end{cases}$$

It suffices to show weak convergence:

$$E\{F[\beta]\} = \lim_{n\to\infty} \{F[x_n(\cdot)]\} \quad \text{for bdd. cont. } F.$$

Let G be continuous and e^{itG} is bdd and continuous, but continuity of characteristic functions then does it

A. T. P.

$$E\{F[\beta]\} = \lim_{k\to\infty} E\{f_k(\beta(\tfrac{1}{k}), \dots, \beta(1))\}$$

polygonalization of β

Suppose $F[\beta] = \int_0^1 \beta(s)\,ds$, then $\lrcorner = \lim_{k\to\infty} E\{\int_0^1 \beta_{\frac{1}{k}}^{(k)}(s)\,ds\}$ in $\frac{1}{k} = \delta_{mes}$

$$= \lim_{k\to\infty}\lim_{n\to\infty} E\{f_k(\tfrac{S_{n_1}}{\sqrt{n}}, \tfrac{S_{n_2}}{\sqrt{n}}, \dots, \tfrac{S_{n_k}}{\sqrt{n}})\} \text{ when } n_j = [\tfrac{j n}{k}]$$

(conjecture $k=n$) $= \lim_{n\to\infty} E\{f_n(\frac{S_1}{\sqrt{n}}, \frac{S_2}{\sqrt{n}}, \dots, \frac{S_n}{\sqrt{n}})\}$ which is our theorem

so you guess the result.

1-dimensional C.L.T.: Let X_1, X_2, be i.i.d.r.v.'s mean 0 variance 1 and $\varphi(x)$ be their common d.f. and $(\alpha, \beta]$ be a half open interval then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \overset{[n]}{\cdots} \int_{-\infty}^{\infty} \chi_{(\alpha,\beta]}(\tfrac{S_n}{\sqrt{n}})\, d\varphi(x_1)\cdots d\varphi(x_n) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \chi_{(\alpha,\beta]} e^{-\frac{u^2}{2}}\, d$$

k-dimensional C.L.T.: Let

$X_{11}, X_{12}, \dots, X_{1n^{(1)}}$ $\quad S_{1,n^{(1)}} = X_{11} + X_{12} + \cdots$

$X_{21}, X_{22}, \dots, X_{2n^{(2)}}$ $\quad \vdots$

the $n^{(j)}$'s depend on n $\quad \cdots$

$X_{k1}, X_{k2}, \dots, X_{kn^{(k)}}$ $\quad S_{k,n^{(k)}} = X_{k1} + X_{k2}$

Let $N = n^{(1)} + \cdots + n^{(k)}$ let $P_N = P\{\alpha_j < S_{j, n^{(j)}} \le \beta_j;\ j = 1, 2, \dots, k\}$,

$$\lim_{n\to\infty} P_N = \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty} \overset{[k]}{\cdots} \int_{-\infty}^{\infty} \chi_{(\alpha,\beta]}(u_1, \dots, u_n)\, e^{-\frac{1}{2}|u|^2}\, du_1\, du_2 \cdots du_k$$

This leads, by routine approximation arguments to

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \overset{[N]}{\cdots} \int_{-\infty}^{\infty} f(\tfrac{S_{1,n^{(1)}}}{\sqrt{n^{(1)}}}, \tfrac{S_{2,n^{(2)}}}{\sqrt{n^{(2)}}}, \dots, \tfrac{S_{k,n^{(k)}}}{\sqrt{n^{(k)}}})\, d\varphi(x_{11}) \cdots d\varphi(x_{kn^{(k)}})$$

$$= \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty} \overset{[k]}{\cdots} \int_{-\infty}^{\infty} f(u_1, \dots, u_k)\, e^{-\frac{1}{2}|u|^2}\, du_1 \cdots du_k \quad \text{for}$$

A.T.P.

Let $u_1 = \sqrt{k}\, v_1$, and $u_j = \sqrt{k}(v_j - v_{j-1})$ $j = 2, 3, \cdots, k$ then

$$= \frac{1}{\sqrt{\left(\frac{2\pi}{k}\right)^k}} \int_{-\infty}^{\infty} \overset{[k]}{\cdots} \int_{-\infty}^{\infty} f(\sqrt{k}\, v_1, \sqrt{k}(v_2 - v_1), \cdots, \sqrt{k}(v_k - v_{k-1})) \cdot$$

$$e^{-\frac{v_1^2}{2 \cdot \frac{1}{k}}} e^{-\frac{(v_2 - v_1)^2}{2 \frac{1}{k}}} \cdots e^{-\frac{(v_k - v_{k-1})^2}{2 \frac{1}{k}}} dv_1 \cdots dv_k$$

$$= E\left\{ g\left(\beta\left(\tfrac{1}{k}\right), \beta\left(\tfrac{2}{k}\right), \cdots, \beta(1)\right) \right\}$$ and the left hand side is

$$= \lim_{n \to \infty} \int_{-\infty}^{\infty} \overset{[N]}{\cdots} \int_{-\infty}^{\infty} g\left(\frac{S_{1,n^{(1)}}}{\sqrt{n^{(1)}}}, \frac{S_{1,n^{(1)}}}{\sqrt{n^{(1)}}} + \frac{S_{2,n^{(2)}}}{\sqrt{n^{(2)}}}, \cdots \right) d\rho(x_{1,1}) \cdots d\rho(x_{k,n^{(k)}})$$

<u>Special Case</u>: Let $n_i = \left[\frac{in}{k}\right]$ and $n^{(j)} = n_j - n_{j-1}$ then we get the result. If $g(t_1, \cdots, t_k)$ is a bdd. Borel measurable function and continuous a.e. on every finite k-dimensional interval then

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \overset{[n]}{\cdots} \int_{-\infty}^{\infty} g\left(\frac{S_{n_1}}{\sqrt{n}}, \frac{S_{n_2}}{\sqrt{n}}, \cdots, \frac{S_{n_k}}{\sqrt{n}} \right) d\rho(x_1) \cdots d\rho(x_k)$$

$$= E\left\{ g\left(\beta\left(\tfrac{1}{k}\right), \beta\left(\tfrac{2}{k}\right), \cdots, \beta(1)\right) \right\}.$$

<u>Kac's Drum</u>: In $\mathbb{R}^2 \supset \Omega$ with $\partial\Omega$ consider $\frac{1}{2}\Delta u + \lambda u = 0$ with $u = 0$ on $\partial\Omega = 0$. $\exists$ a discrete spectrum $\lambda_1, \cdots$ and $u_1(x, y), \cdots$ the normalized eigenfunctions. Consider

$$\sum_{\lambda_i < \lambda} 1 = \#\text{ of eigenvalues} < \lambda$$

This is an increasing function of λ.

Herman Weyl: $\sum_{\lambda_j < \lambda} 1 \sim \frac{|\Omega| \lambda}{2\pi}$ as $\lambda \to \infty$.

Carlemann showed $\sum u_j^2(x, y) \sim \frac{\lambda}{2\pi}$ $\forall x, y \in \Omega$ as $\lambda \to \infty$.

A.T.P.

Start a B.M. at (x_0, y_0) and $p(x_0, y_0; x, y, t)$ be the p.d.f. of a 2-d B.M. starting at (x_0, y_0) reaching (x, y) in time t without crossing $\partial\Omega$.

Einstein-Smoluchowski: $p(x_0, y_0; x, y, t)$ solves
$$\begin{cases} \frac{\partial p}{\partial t} = \frac{1}{2}\Delta p \text{ in } \Omega \\ p = 0 \text{ on } \partial\Omega \ \forall t \\ \int g(x,y)\, p(x_0, y_0; x, y, t) \\ \quad = g(x_0, y_0) \text{ as } t \to 0 \end{cases}$$

Assume $p = T(t)\,U(x,y)$ gives
$$T'U = \frac{T}{2}\Delta U \qquad U = 0 \text{ on } \partial\Omega \ \forall t$$
$$\frac{T'}{T} = \frac{\Delta U}{2U} = -\lambda \quad \text{yields}$$

So $\left.\begin{matrix} T(t) = e^{-\lambda_j t} \\ U(x,y) = u_j(x,y) \end{matrix}\right\}$ and U is an eigenfunction of the Laplacian, ie $j = 1, 2, \dots$

So $p(x_0, y_0; x, y, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j(x_0, y_0)\, u_j(x, y)$, and we know
$$p(x_0, y_0; x_0, y_0, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0, y_0).$$

Let $p^*(x_0, y_0; x, y, t)$ be the p.d.f. for unrestricted B.M. starting at (x_0, y_0) getting to (x, y) at t
$$p^*(x_0, y_0; x, y, t) = \frac{1}{2\pi t} e^{-\frac{(x-x_0)^2}{2t} - \frac{(y-y_0)^2}{2t}}.$$

$$\therefore \sum_{j=1}^{\infty} e^{-\lambda_j} u_j^2(x_0, y_0) \sim \frac{1}{2\pi t} \quad \text{as } t \to 0.$$

<u>Karamata</u> <u>Tauberian</u> <u>Theorem</u>: $\int_0^\infty e^{-\lambda t}\, d\alpha(\lambda)$ and let $\alpha(\lambda)$ be non-decreasing on $(0, \infty)$ and assume this Laplace-Stitje's transform exists, then if $f(t) = \int_0^\infty e^{-\lambda t} d\alpha(\lambda)$ is asymptotic to $At^{-\gamma}$ as $t \to 0$ where γ and A are constants, then

A.T.P.

$\alpha(\lambda) \sim \dfrac{A\lambda^{\gamma}}{\Gamma(\gamma+1)}$ as $\lambda \to \infty$ $(\lambda \to 0)$.

$$\int_0^\infty e^{-\lambda t}\, d\alpha(\lambda) = \sum_{i=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0,y_0) \quad \text{where} \quad \alpha(\lambda) = \sum_{\lambda_j < \lambda} u_j^2(x_0,y_0)$$

by K-T theorem $\alpha(\lambda) \sim \dfrac{\lambda}{2\pi}$ as $\lambda \to \infty$. Integrating over Ω we get Weyl's theorem.

A.T.P.

Karamata Tauberian Theorem: Let $\alpha(t)$ be nondecreasing such that $f(s) = \int_0^\infty e^{-st}\, d\alpha(t)$ converges for $s>0$. If for some $\gamma > 0$

$$f(s) \sim A s^{-\gamma} \quad s \to 0^+,\ (s\to\infty) \text{ then}$$

$$\alpha(t) \sim A t^\gamma / \Gamma(\gamma+1) \quad t\to\infty,\ (t\to 0^+).$$

Note: 1. Chose $\alpha(t) = t^\gamma$, then $\alpha'(t) = \gamma t^{\gamma-1}$ then

$$f(s) = \gamma\int_0^\infty e^{-st} t^{\gamma-1} dt = \frac{\gamma\Gamma(\gamma)}{s^\gamma} = \frac{\Gamma(\gamma+1)}{s^\gamma}$$

2. If $\alpha(t) = P\{X \le t\}$ and $f(s) = E\{e^{-sX}\}$

$$= \int_0^\infty e^{-st} dP\{X \le t\}.$$

Proof: (Lemma) Let g be Riemann integrable on $[0,1]$ and let $\gamma > 0$. For any $\varepsilon > 0$ $\exists$ polynomials $p(x)$ and $P(x)$ $\ni$ $p(x) < g(x) < P(x)$ $x \in [0,1]$ and $\int_0^\infty e^{-t} t^{\gamma-1}[P(e^{-t}) - p(e^{-t})]dt < \varepsilon$.

(Proof:) Consider g the indicator function of (α,β) when $(\alpha,\beta) \subset [0,1]$. Let $\eta > 0$ be given. $\exists$ a continuous function $h(x)$ $\ni$ $h(x) \ge g(x)$ $x\in[0,1]$ and $\int_0^\infty e^{-t} t^{\gamma-1}[h(e^{-t}) - g(e^{-t})]dt < \eta$

By the W-approximation theorem $\exists$ a polynomial Q $\ni$

$$|Q(x) - h(x)| < \eta \quad \forall x \in [0,1]$$

Let $P(x) = Q(x) + \eta$ then $g(x) \le h(x) < P(x)$.

$$\text{Now } \int_0^\infty e^{-t} t^{\gamma-1}[P(e^{-t}) - g(e^{-t})]dt \le \int_0^\infty e^{-t}t^{\gamma-1}[Q(e^{-t}) - h(e^{-t})]dt$$

$$+ \int_0^\infty e^{-t} t^{\gamma-1}[h(e^{-t}) - g(e^{-t})]dt$$

$$+ \eta \int_0^\infty e^{-t} t^{\gamma-1} dt$$

$$< \eta + 2\eta \int_0^\infty e^{-t} t^{\gamma-1} dt.$$

A.T.P.

we can also get $p(x) \le g(x)$ in an analogous manner $\ni$

$$\int_0^\infty e^{-t} t^{\gamma-1}[g(e^{-t}) - p(e^{-t})]dt < \eta + 2\eta \int_0^\infty e^{-t} t^{\gamma-1} dt.$$

So the lemma is true for indicator functions of intervals in $[0,1]$. Therefore the lemma is also true for step functions on $[0,1]$ having a finite number of jumps.

Now let g be R.I. on $[0,1]$. Let $M = \sup_{0 \le x \le 1} |g(x)|$. Choose $\delta > 0$ and $R > 0$ $\ni$ $2M\left[\int_0^\delta e^{-t} t^{\gamma-1} dt + \int_R^\infty e^{-t} t^{\gamma-1} dt\right] < \frac{\varepsilon}{6}$. Since g is R.I. on $[0,1]$ $\exists$ step functions g_1 and g_2 on $[\delta, R]$ each with a finite number of jumps $\ni$ $g_1(e^{-t}) \le g(e^{-t}) \le g_2(e^{-t})$ ⊛ $t \in [\delta, R]$ and

$$\int_\delta^R [g_2(e^{-t}) - g_1(e^{-t})] e^{-t} t^{\gamma-1} dt < \frac{\varepsilon}{6}.$$ Now let

$$g_2(e^{-t}) = M \qquad 0 \le t < \delta \text{ and } R < t < \infty$$
$$g_1(e^{-t}) = M \qquad \text{" } \qquad \text{"}$$
, then ⊛ holds everywhere. By the first part of the proof $\exists$ polynomials $p(x)$ and $P(x)$ $\ni$

$$p(x) < g_1(x) \le g(x) \le g_2(x) < P(x) \qquad x \in [0,1]$$

$$\ni \int_0^\infty e^{-t} t^{\gamma-1}[P(e^{-t}) - g_2(e^{-t})]dt < \frac{\varepsilon}{3}$$

$$\int_0^\infty e^{-t} t^{\gamma+1}[g_1(e^{-t}) - p(e^{-t})]dt < \frac{\varepsilon}{3}$$

Th

A.T.P.

$$\int_0^\infty e^{-t}t^{\gamma-1}\left[g_2(e^{-t})-g_1(e^{-t})\right]dt=\int_0^\delta+\int_\delta^R+\int_R^\infty<\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{\varepsilon}{3}$$

and finally $\int_0^\infty e^{-t}t^{\gamma-1}\left[P(e^{-t})-p(e^{-t})\right]dt$

$$=\int_0^\infty e^{-t}t^{\gamma-1}\left[\underbrace{P(e^{-t})-g_2(e^{-t})}_{\frac{\varepsilon}{3}}+\underbrace{g_2(e^{-t})-g_1(e^{-t})}_{\frac{\varepsilon}{3}}+\underbrace{g_1(e^{-t})-p(e^{-t})}_{\frac{\varepsilon}{3}}\right]dt$$

$$\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

(Theorem.) Let $\alpha(t)$ be nondecreasing and $\ni \int_0^\infty e^{-st}d\alpha(t)=f(s)$ converges for all $s>0$. Assume for some $\gamma>0$ $f(s)\sim\frac{1}{s^\gamma}$ $s\to 0^+$ $(s\to\infty)$, let $g(x)$ be of bdd variation on $[0,1]$ then:

$$(1)\quad \int_0^\infty e^{-st}g(e^{-st})\,d\alpha(t)\sim\frac{1}{s^\gamma}\frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-t}g(e^{-t})t^{\gamma-1}dt$$

as $s\to 0^+$ $(s\to\infty)$.

(Proof). Let $\alpha(t)$ be continuous except perhaps at $x_0, x_1, \cdots$, and let $g(e^{-t})$ be continuous except perhaps at $y_0, y_1, \cdots$, and let E be the set of point y_i/x_j $i,j=0,1,2,\cdots$. If $s\notin E$ the integral on the l.h.s. of (1) exists. Since E is countable, E^c is dense in $(0,\infty)$ and $\therefore$ we have no problem in letting $s\to 0^+$ or $s\to\infty$ through points in E^c.

Let $\varepsilon>0$ be given. From the lemma $\exists$ $p(x)$ and $P(x)$ $\ni$

$$p(x)<g(x)<P(x)\qquad x\in[0,1]\quad\ni$$

$$(2)\quad \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-t}t^{\gamma-1}\left[P(e^{-t})-p(e^{-t})\right]dt<\varepsilon.$$

Now
$$(3)\quad \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-t}t^{\gamma-1}p(e^{-t})\,dt\leq\frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-t}t^{\gamma-1}g(e^{-t})\,dt\leq\frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-t}t^{\gamma-1}P(e^{-t})\,dt.$$

and α is non-decreasing so

$$(4)\quad \int_0^\infty e^{-st}p(e^{-st})\,d\alpha(t)\leq\int_0^\infty e^{-st}g(e^{-st})\,d\alpha(t)\leq\int_0^\infty e^{-st}P(e^{-st})$$

A.T.P.

In the hypothesis we had $f(s) \sim \frac{1}{s^\gamma}$ as $s \to 0^+$ $(s \to \infty)$. Replace s by $(n+1)s$ where $n \in \mathbb{Z}^+$ we get

$$\int_0^\infty e^{-st} e^{-nst} \, d\alpha(t) \sim \frac{1}{(n+1)^\gamma s^\gamma} = \frac{1}{\Gamma(\gamma)} \frac{1}{s^\gamma} \int_0^\infty e^{-t} e^{-nt} t^{\gamma-1} \, dt$$
$$s \to 0^+ \quad (s \to \infty)$$

It follows then that

$$\int_0^\infty e^{-st} Q(e^{-st}) \, d\alpha(t) \sim \frac{1}{\Gamma(\gamma)} \frac{1}{s^\gamma} \int_0^\infty e^{-t} t^{\gamma-1} Q(e^{-t}) \, dt.$$

Multiply through (4) by s^γ and let $s \to 0^+$ $(s \to 0)$ though values of E^c. So we get

$$\frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-t} t^{\gamma-1} p(e^{-t}) \, dt \le \varliminf_{\substack{s \to 0^+ \\ (s \to \infty)}} {}_{s \in E^c} \, s^\gamma \int_0^\infty e^{-st} g(e^{-st}) \, d\alpha(t)$$

$$\le \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-t} t^{\gamma-1} P(e^{-t}) \, dt$$

$\therefore$ the limit exists from (3) and (2) and the fact that $\varepsilon > 0$ is arbitrary

$$\lim_{\substack{s \to 0^+ \\ (s \to \infty)}} {}_{s \in E^c} \, s^\gamma \int_0^\infty e^{-st} g(e^{-st}) \, d\alpha(t) = \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-t} g(e^{-t}) \, dt.$$

<u>Karamata Theorem is a Corollary</u>:

<u>Proof</u>: Let $g(x) = \begin{cases} 0 & 0 \le x < \frac{1}{e} \\ \frac{1}{x} & \frac{1}{e} \le x < 1 \end{cases}$, ie $g(e^{-t}) = \begin{cases} e^t & 0 \le t \le 1 \\ 0 & 1 < t < \infty \end{cases}$

or $g(e^{-st}) = \begin{cases} e^{st} & 0 \le t < \frac{1}{s} \\ 0 & \frac{1}{s} \le t < \infty \end{cases}$. So (1) becomes

$\lim_{\substack{s \to 0^+ \\ (s \to \infty)}} s^{\gamma} \alpha(\tfrac{1}{s}) = \frac{1}{\Gamma(\gamma+1)}$. But α is non-decreasing so we let $s = \frac{1}{t}$ we have $\alpha(t) \sim \frac{1}{\Gamma(\gamma+1)} t^{\gamma}$ as $t \to \infty$, $(t \to 0^+)$.

E'3s

<u>Potential Theory</u>: In $\mathbb{R}^3$, Let Ω be a bdd closed region, and C is the space of all continuous functions $\vec{r}(\tau)$ starting from the origin. Let $\chi_\Omega(\vec{r})$ be the indicator function of Ω and consider the functional on C given by

$$T_\Omega(\vec{y}, \vec{r}(\cdot)) = \int_0^\infty \chi_\Omega(\vec{y} + \vec{r}(\tau))\, d\tau \qquad \vec{y} \in \mathbb{R}^3$$

= total occupation time of $\vec{r}(\cdot)$ in Ω, translated by $\vec{y}$. (3d B.M. starting at $\vec{y}$)

Impose on C the Wiener measure. Consider

$$E\{T_\Omega(\vec{y}, \vec{r}(\cdot))\} = \int_0^\infty P\{\vec{y} + \vec{r}(\tau) \in \Omega\}\, d\tau$$

by Fubini's theorem we get

but $P\{\vec{y} + \vec{r}(\tau) \in \Omega\} = \frac{1}{(2\pi\tau)^{3/2}} \int_\Omega e^{-\frac{|\vec{r}-\vec{y}|^2}{2\tau}}\, d\vec{r}$ so that

$$E\{T_\Omega(\vec{y}, \vec{r}(\cdot))\} = \int_\Omega d\vec{r} \int_0^\infty \frac{1}{(2\pi\tau)^{3/2}} e^{-\frac{|\vec{r}-\vec{y}|^2}{2\tau}}\, d\tau$$

here we see why we are in $\mathbb{R}^3$!

this is convergent in $\mathbb{R}^3$.

$$= \frac{1}{2\pi} \int_\Omega \frac{d\vec{r}}{|\vec{r}-\vec{y}|} < \infty$$

We see that in $\mathbb{R}^3$ a.e. path starting from $\vec{y}$ spends a finite time in Ω.

We see now the the k^{th} moment of the occupation time is

$$E\{T_\Omega^k(\vec{y},\vec{r}(\cdot))\} = \frac{k!}{(2\pi)^k}\int_\Omega \overset{[k]}{\cdots}\int_\Omega \frac{d\vec{r}_1}{|\vec{r}_1-\vec{y}|}\frac{d\vec{r}_2}{|\vec{r}_2-\vec{r}_1|}\cdots\frac{d\vec{r}_k}{|\vec{r}_k-\vec{r}_{k-1}|} \qquad k=1,2,\cdots.$$

We examine $k=2$:

$$E\{T_\Omega^2(\vec{y},\vec{r}(\cdot))\} = \int_0^\infty\int_0^\infty P\{\vec{y}+\vec{r}(\tau_1)\in\Omega\}P\{\vec{y}+\vec{r}(\tau_2)\in\Omega\}\,d\tau_1 d\tau_2$$

$$= 2\iint_{0\le\tau_1<\tau_2<\infty} d\tau_1 d\tau_2 \int_\Omega\int_\Omega \frac{1}{(2\pi\tau_1)^{3/2}} e^{-\frac{|\vec{r}_1-\vec{y}|}{2\tau_1}} \frac{1}{[2\pi(\tau_2-\tau_1)]^{3/2}} e^{-\frac{|\vec{r}_2-\vec{r}_1|}{2(\tau_2-\tau_1)}} d\vec{r}_1 d\vec{r}_2$$

$$= \frac{2}{(2\pi)^2}\int_\Omega\int_\Omega \frac{1}{|\vec{r}_1-\vec{y}|}\frac{1}{|\vec{r}_2-\vec{r}_1|}\,d\vec{r}_1 d\vec{r}_2,$$ etc. The formula

or the k^{th} moment suggests we consider the eigenvalue problem.

$$\frac{1}{2\pi}\int_\Omega \frac{\varphi(\vec{\rho})}{|\vec{r}-\vec{\rho}|}d\vec{\rho} = \lambda\varphi(\vec{r}) \qquad \vec{r}\in\Omega.$$ This kernel

is Hilbert-Schmidt as $\iint_{\Omega\,\Omega}\frac{1}{|\vec{r}-\vec{\rho}|^2}d\vec{r}\,d\vec{\rho}<\infty$ and will will now show it is positive definite:

$$\circledast\ \frac{1}{2\pi}\int_\Omega\int_\Omega \frac{\varphi(\vec{r})\varphi(\vec{\rho})}{|\vec{r}-\vec{\rho}|}d\vec{r}\,d\vec{\rho} > 0 \quad \forall\,\varphi(\vec{r})\neq 0 \text{ in } L^2(\Omega$$

Note $\frac{1}{2\pi}\frac{1}{|\vec{r}-\vec{\rho}|} = \int_0^\infty \frac{1}{(2\pi\tau)^{3/2}} e^{-\frac{|\vec{r}-\vec{\rho}|^2}{2\tau}}d\tau = \int_0^\infty d\tau \frac{1}{(2\pi\tau)^{3/2}}\frac{\tau^{3/2}}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{i\vec{\xi}\cdot(\vec{r}-\vec{\rho})} e^{-\frac{|\xi|^2\tau}{2}}d\vec{\xi}$

$$= \frac{1}{(2\pi)^3}\int_0^\infty d\tau\int_{\mathbb{R}^3} d\vec{\xi}\, e^{i\vec{\xi}\cdot(\vec{r}-\vec{\rho})} e^{-\frac{|\xi|^2\tau}{2}}$$ so

$$\circledast = \frac{1}{(2\pi)^3}\int_0^\infty d\tau\int_{\mathbb{R}^3} d\vec{\xi}\, e^{-\frac{|\xi|^2\tau}{2}}\left|\int_\Omega \varphi(\vec{\rho})e^{i\vec{\xi}\cdot\vec{\rho}}d\vec{\rho}\right|^2 > 0 \quad \forall\,\varphi(\vec{r})\neq 0$$

in $L^2(\Omega$

Therefore we have $\lambda_1, \lambda_2, \cdots$ discrete positive spectrum and a complete orthonormal basis for $L^2(\Omega)$ in the eigenfunctions.

<u>Lemma:</u> $$\frac{1}{k!} E\left\{T_\Omega^k(\vec{y}, \vec{r}(\cdot))\right\} = \sum_{j=1}^{\infty} \lambda_j^{k-1} \int_\Omega \varphi_j(\vec{r})\, d\vec{r}\, \underbrace{\frac{1}{2\pi}\int_\Omega \frac{\varphi_j(\vec{\rho})\, d\vec{\rho}}{|\vec{\rho}-\vec{y}|}}_{\lambda_j \varphi_j(\vec{y})}$$

$\forall y \in \mathbb{R}^3$, and if $y \in \Omega$ $\qquad \lambda_j \varphi_j(\vec{y})$.

<u>Proof:</u> $$\frac{1}{k!} E\left\{T_\Omega^k(\vec{y}, \vec{r}(\cdot))\right\} \overset{\text{by Mercer's Theorem}}{=} \frac{1}{2\pi}\int_\Omega \frac{1}{|\vec{r}_1-\vec{y}|} \int_\Omega \overset{(k-1)}{\cdots} \int_\Omega \left(\sum_{j_1=1}^{\infty} \lambda_{j_1} \varphi_{j_1}(\vec{r}_1)\varphi_{j_1}(\vec{r}_2)\right)$$

$$\cdots\left(\sum_{j_{k-1}}^{\infty} \lambda_{j_{k-1}} \varphi_{j_{k-1}}(\vec{r}_k)\varphi_{j_{k-1}}(\vec{r}_{k-1})\right) d\vec{r}_1 \cdots d\vec{r}_k$$

$$= \frac{1}{2\pi}\int_\Omega \frac{1}{|\vec{r}_1-\vec{y}|} \int_\Omega \sum_{j_1=1}^{\infty} \lambda_{j_1}^{k-1} \varphi_{j_1}(\vec{r}_1)\varphi_{j_1}(\vec{r}_k)\, d\vec{r}_k\, d\vec{r}_1 .$$

Consider the moment generating function, with $z \in \mathbb{C}$

$$E\left\{e^{z T_\Omega(\vec{y}, \vec{r}(\cdot))}\right\} = \sum_{k=0}^{\infty} \frac{z^k}{k!} E\left\{T_\Omega^k(\vec{y}, r(\cdot))\right\}$$

$$= 1 + \frac{z}{2\pi}\sum_{j=1}^{\infty}\left(\frac{1}{1-\lambda_j z}\right)\int_\Omega \varphi_j(\vec{r})\, d\vec{r} \int_\Omega \frac{\varphi_j(\vec{\rho})}{|\vec{\rho}-\vec{y}|}\, d\vec{\rho}$$

This series converges if $|z|$ is small enough $\left(< \frac{1}{\lambda_{max}}\right)$. The left hand side is analytic if $Re\{z\} < 0$ since $T_\Omega \geq 0$. The r.h.s. is analytic for $Re\{z\} < 0$ so by analytic continuation this identity holds with $Re\{z\} < 0$.

Review: Ω closed bdd. set in $\mathbb{R}^3$ and we have

$$T_\Omega = \int_0^\infty \chi_\Omega(\vec{y} + \vec{r}(\tau))\,d\tau \text{ , with Wiener measure on}$$

$\cdot) \in C$ with $\vec{r}(0) = 0$. (Total occupation time in Ω starting at $\vec{y}$).

$$E\{T_\Omega(\vec{y}, \vec{r}(\cdot))\} = \frac{1}{2\pi}\int_\Omega \frac{d\vec{r}}{|\vec{r}-\vec{y}|} < \infty$$ and therefore

all paths starting at $\vec{y}$ spend at most a finite time in Ω. Also we have

$$E\{T_\Omega^k(\vec{y}, \vec{r}(\cdot))\} = \frac{k!}{(2\pi)^k}\int_\Omega \overset{[k]}{\cdots}\int_\Omega \frac{d\vec{r}_1}{|\vec{r}_1-\vec{y}|}\frac{d\vec{r}_2}{|\vec{r}_2-\vec{r}_1|}\cdots\frac{d\vec{r}_k}{|\vec{r}_k-\vec{r}_{k-1}|} \text{ , so}$$

Let $\lambda_1, \lambda_2, \dots$ and $\varphi_1, \varphi_2, \dots$ be the eigenvalues and normalized eigenfunctions of

$$\frac{1}{2\pi}\int \frac{\varphi(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{r}|} = \lambda\varphi(\vec{r}) \text{ , } \vec{r} \in \Omega \text{, and also}$$

$$\frac{1}{k!}E\{T_\Omega^k(\vec{y}, \vec{r}(\cdot))\} = \sum_{j=1}^\infty \lambda_j^{k-1}\int_\Omega \varphi_j(\vec{r})\,d\vec{r}\ \frac{1}{2\pi}\int_\Omega \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \quad \vec{y} \in \mathbb{R}^3$$

$$= \sum_{j=1}^\infty \lambda_j^k \int_\Omega \varphi_j(\vec{r})\varphi_j(\vec{y})\,d\vec{r} \quad y \in \Omega.$$

Let $u > 0$ and define

$$h(\vec{y}, \vec{u}) = E\{e^{-uT_\Omega(\vec{y}, \vec{r}(\cdot))}\} =$$

$$(1) \qquad 1 - \frac{u}{2\pi}\sum_{j=1}^\infty\left(\frac{1}{1+\lambda_j u}\right)\int_\Omega \varphi_j(\vec{r})\,d\vec{r}\int_\Omega \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|}$$

Note: The series converges uniformly on compact sets (in $\mathbb{C}$) because

$$\frac{1}{1+\lambda_j u} < 1 \quad \text{and} \left(\sum_{j=1}^\infty \int_\Omega \varphi_j(\vec{r})\,d\vec{r}\int_\Omega \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|}\right)^2 \le$$

A.T.P.

by Parseval

$$\le \sum_{j=1}^{\infty}\Big(\int_\Omega \varphi_j(\vec r)\,d\vec r\Big)^2 \sum_{j=1}^{\infty}\Big(\int_\Omega \frac{\varphi_j(\vec\rho)\,d\vec\rho}{|\vec\rho - y|}\Big)^2 = |\Omega| \int_\Omega \frac{d\vec\rho}{|\vec\rho-\vec y|^2} < \infty .$$

So we have uniform convergence by Weierstrass. M-test & hence analyticity.

If $\vec y \in \Omega$ this last becomes $h(\vec y,\vec u) = 1 - \sum_{j=1}^{\infty} \frac{\lambda_j u}{1+\lambda_j u}\int_\Omega \varphi_j(\vec r)\,d\vec r\, \varphi_j(\vec y)$, so

$$\frac{1}{2\pi}\int_\Omega \frac{h(\vec y,u)}{|\vec y-\vec r|}\,d\vec y = \frac{1}{2\pi}\int_\Omega \frac{d\vec y}{|\vec y - \vec r|} - \sum_{j=1}^{\infty}\frac{\lambda_j u}{1+\lambda_j u}\int_\Omega \varphi_j(\vec\rho)\,d\vec\rho\, \frac{1}{2\pi}\int \frac{\varphi_j(\vec y)\,d\vec y}{|\vec y-\vec r|}$$

but $\displaystyle \frac{1}{2\pi}\int_\Omega \frac{d\vec y}{|\vec y-\vec r|} = \sum_{j=1}^{\infty}\int_\Omega \varphi_j(\vec\rho)\,d\vec\rho\, \frac{1}{2\pi}\int \frac{\varphi_j(\vec y)\,d\vec y}{|\vec y-\vec r|}$. Thus we

have
$$\frac{1}{2\pi}\int_\Omega \frac{h(\vec y,u)\,d\vec y}{|\vec y-\vec r|} = \sum_{j=1}^{\infty}\frac{1}{1+\lambda_j u}\int_\Omega \varphi_j(\vec\rho)\,d\vec\rho\,\frac{1}{2\pi}\int_\Omega \frac{\varphi_j(\vec y)\,d\vec y}{|\vec y-\vec r|}$$

$$\therefore\ \frac{1}{2\pi}\int_\Omega \frac{h(\vec y,u)\,d\vec y}{|\vec y-\vec r|} = \frac{1}{u}\big(1-h(\vec r,u)\big) \qquad \forall\, \vec r\in \mathbb{R}^3$$, or

renaming the variables and any $\vec y\in\mathbb{R}^3$

$$\frac{1}{2\pi}\int_\Omega \frac{h(\vec\rho,u)}{|\vec y-\vec\rho|}\,d\vec\rho = \frac{1}{u}\big(1-h(\vec y,u)\big). \qquad (2)$$

Observations:

If $\vec y\notin\Omega$, $h(\vec y,u)$ is harmonic since (1) shows each term in the series is harmonic in $\vec y$, and the series converges uniformly on compact Ω's. Moreover, from (1):

$$h(\vec y,u) > 1 - \frac{u}{2\pi}\Big\{\sum_{j=1}^{\infty}\Big(\int_\Omega \varphi_j(\vec\rho)\,d\vec\rho\Big)^2\Big\}^{1/2}\Big\{\sum_{j=1}^{\infty}\Big(\int_\Omega \frac{\varphi_j(\vec\rho)}{|\vec\rho-\vec y|}\,d\vec\rho\Big)^2\Big\}^{1/2}$$

$$> 1 - \frac{u}{2\pi}|\Omega|^{1/2}\Big(\int_\Omega \frac{d\vec\rho}{|\vec\rho-\vec y|^2}\Big)^{1/2}$$ and as

$0\le h(\vec y,u)\le 1$ and so $\displaystyle\lim_{|\vec y|\to\infty} h(\vec y,u) = 1.$ (3')

A. T. P.

And for $\vec{y} \in \Omega$ $\quad \Delta\left(\int_\Omega \frac{h(\vec{y},u)\,d\vec{y}}{|\vec{\rho}-\vec{y}|}\right) = -4\pi h(\vec{y},u).$
Courant - Hilbert II 245-246.

So applying the Laplacian to both sides of (2) we get:

$$-2\ h(\vec{y},u) = -\frac{1}{u}\Delta h(\vec{y},u) \quad \text{or} \quad \text{we have}$$

$$\tfrac{1}{2}\Delta h(\vec{y},u) - uh = 0. \quad (3)' \quad (y \in \Omega).$$

Consider $V(\vec{y}) = \lim\limits_{u\to\infty}(1-h(\vec{y},u)) = P\{T_\Omega(\vec{y},\vec{r}(\cdot)) > 0\}$, this
↑ capacitory potential.
is elementary from the definition of the moment generating function.
Example: Let Ω be a sphere of radius centered at 0. It is clear that in this special case $h(\vec{y},u)$ is spherically symmetric, one sees this from the moments of $T_\Omega(\vec{y},\vec{r}(\cdot))$. Since, as we have seen above, $h(\vec{y},u)$ is harmonic outside Ω, we have that

$$h(\vec{y},u) = \frac{\alpha(u)}{|\vec{y}|} + \beta(u) \qquad \vec{y} \notin \Omega.$$

But since , from (3') we see $\beta(u) = 1$, ie $h(\vec{y},u) = \frac{\alpha(u)}{|\vec{y}|} + 1 \ \ \vec{y} \in \Omega.$

But from (3) for $y \in \Omega$ $\quad h(\vec{y},u) = \gamma(u)\frac{\sinh(\sqrt{2u}\,|\vec{y}|)}{|\vec{y}|}$, subsituting this into the equation

$$\frac{1}{2\pi}\int_\Omega \frac{h(\vec{\rho},u)\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} = \frac{1}{u}(1-h(\vec{y},u)) \quad \text{we get}$$

$$\gamma(u) = \frac{1}{\sqrt{2u}\cosh(a\sqrt{2u})}$$. Now $h(\vec{y},u)$ is continuous $\forall \vec{y}$, from the uniform convergence of that series, therefore

$\frac{1}{\sqrt{2u}} \frac{\sinh(\sqrt{2u}\, a)}{\cosh(a\sqrt{2u})} \frac{1}{a} = \frac{\alpha(u)}{a} + 1$. Thus we finally get

$$h(\vec{y},u) = \begin{cases} 1 - \frac{a}{|\vec{y}|}\left(1 - \frac{\tanh(a\sqrt{2u})}{a\sqrt{2u}}\right) & \vec{y} \notin S(0,a) \\ \frac{\sinh(\sqrt{2u}\,|\vec{y}|)}{\sqrt{2u}\cosh(\sqrt{2u}\,a)\,|\vec{y}|} & \vec{y} \in S(0,a) \end{cases}$$

$$\therefore \quad U(\vec{y}) = \lim_{u\to\infty}(1 - h(\vec{y},u)) = P\{T_{S(0,a)}(\vec{y},\vec{r}(\cdot)) > 0\}$$

$$= \begin{cases} \frac{a}{|\vec{y}|} & y \notin S(0,a) \\ 1 & y \in S(0,a) \end{cases}$$

. Thus this is the capacitory potential of $S(0,a)$, a very well known fact.

Back to the general case, we have $\forall \vec{y} \in \mathbb{R}^3$

$$1 - E\{e^{-u T_\Omega(\vec{y},\vec{r}(\cdot))}\} = \sum_{j=1}^{\infty} \frac{1}{\lambda_j + \frac{1}{u}} \int_\Omega \varphi_j(\vec{r})\,d\vec{r}\ \frac{1}{2\pi}\int_\Omega \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|},$$

as noted before, $0 \le 1 - h(\vec{y},u) \le 1$, and moreover,

$$1 - h(\vec{y},u_1) \le 1 - h(\vec{y},u_2) \quad \text{if} \quad u_1 < u_2 .$$

Since

$$0 \le e^{-u T_\Omega(\vec{y},\vec{r}(\cdot))} \le 1 \quad \text{and since} \quad \lim_{u\to\infty} e^{-u T_\Omega(\vec{y},\vec{r}(\cdot))} = \begin{cases} 0 & T_\Omega > \\ 1 & T_\Omega = 0 \end{cases}$$

we have from the bounded convergence theorem

$$U(\vec{y}) = \lim_{u\to\infty}(1 - \bar{h}(\vec{y},u)) = P\{T_\Omega(\vec{y},\vec{r}(\cdot)) > 0\}, \text{ hence also}$$

$$U(\vec{y}) = \lim_{u\to\infty} \sum_{j=1}^{\infty} \frac{1}{\frac{1}{u} + \lambda_j} \int_\Omega \varphi_j(\vec{r})\,d\vec{r}\ \frac{1}{2\pi}\int_\Omega \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \quad \text{and this}$$

holds $\forall y \in \mathbb{R}^3$.

Case 1: Consider 1st $\vec{y} \in \mathring{\Omega}$.

Clearly, the continuity of the paths $\vec{r}$ immediatly implies
$$V(\vec{y}) = P\{T_\Omega(\vec{y}, \vec{r}(\cdot)) > 0\} = 1.$$

Remark on the side: Let $\vec{y} \in \Omega^\circ$ then we have $V(\vec{y}) = 1$, so
$$1 = \lim_{u\to\infty} \sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j + \frac{1}{u}} \int_\Omega \varphi_j(\vec{r})\varphi_j(\vec{y})\,d\vec{r}$$, this is a summability result.

Case 2: $\vec{y} \notin \Omega$, we have already seen that the function $1-h(\vec{y},u)$ are harmonic in $\vec{y}$ and we noted they are monotone dercasing in u and moreover $\lim_{u\to\infty}(1-h(\vec{y},u)) = V(\vec{y}) = P\{T_\Omega(\vec{y}, \vec{r}(\cdot)) > 0\}$ exists,

therefore by Harnack's theorem $V(\vec{y})$ is harmonic for $\vec{y} \notin \Omega$.

Let $S(0,a)$ be a sphere of radius a centered at 0 which contains Ω.

$$P\{T_\Omega(\vec{y}, \vec{r}(\cdot)) > 0\} \le P\{T_{S(0,a)}(\vec{y}, \vec{r}(\cdot)) > 0\}.$$ From the example this last prob.

$$P\{T_\Omega(\vec{y}, \vec{r}(\cdot)) > 0\} \le \frac{a}{|\vec{y}|} \qquad y \notin S(0,a).$$ So

$$\lim_{|\vec{y}|\to\infty} V(\vec{y}) = 0.$$

Case 3: $\vec{y}_0 \in \partial\Omega$, and assume it is regular in the sense of Poincaré, ie, $\exists$ a sphere $S(\vec{y}_{*0}, \varepsilon)$ which lies entirely in Ω and contains $\vec{y}_0$.
We have first for $\vec{y} \notin \Omega$
$$V(\vec{y}) = P\{T_\Omega(\vec{y}, \vec{r}(\cdot)) > 0\} \ge P\{T_{S(\vec{y}_{*0},\varepsilon)}(\vec{y}, \vec{r}(\cdot)) > 0\}.$$

$= \frac{\varepsilon}{|\vec{y}-\vec{y}_*|}$. As $\vec{y} \to \vec{y}_0$, $\vec{y} \notin \Omega$ $\frac{\varepsilon}{|\vec{y}-\vec{y}_*|} \to \frac{\varepsilon}{|\vec{y}_0-\vec{y}_*|} = 1$, and since $V(\vec{y}) \le 1$ we have $\lim_{\vec{y}\to\vec{y}_0} V(\vec{y}) = 1$.

Hence we see that if Ω is a bounded closed region, each point on the boundary of which is regular in the sense of Poincaré, then $V(\vec{y})$ is the capacitory potential of Ω. Remember

$$V(\vec{y}) = \lim_{\delta\to 0} \sum_{j=1}^{\infty} \frac{1}{\lambda_j+\delta} \int_\Omega \varphi_j(\vec{r})\,d\vec{r}\,\frac{1}{2\pi}\int_\Omega \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|}.$$

$\forall \vec{y} \in \mathbb{R}^3$ $1-h(\vec{y},u) = \frac{1}{2\pi}\int_\Omega \frac{u h(\vec{\rho},u)\,d\vec{\rho}}{|\vec{y}-\vec{\rho}|}$. We note this implies

$$\lim_{|\vec{y}|\to\infty} |\vec{y}|(1-h(\vec{y},u)) = \frac{1}{2\pi}\int_\Omega u h(\vec{\rho},u)\,d\vec{\rho}$$, again $\Omega \subset S(0,a)$

then $h(\vec{y},u) = E\{e^{-uT_\Omega}\} \ge E\{e^{-uT_{S(0,a)}}\}$ therefore for

$\vec{y} \notin S(0,a)$ $h(\vec{y},u) \ge 1 - \frac{a}{|\vec{y}|}$ or $1-h(\vec{y},u) \le \frac{a}{|\vec{y}|}$ and so

$\frac{u}{2\pi}\int_\Omega h(\vec{\rho},u)\,d\vec{\rho} \le a$. Consider now the family of set functions $\mu_u(B)$ defined by

$$\mu_u(B) = \frac{u}{2\pi}\int_{B\cap\Omega} h(\vec{\rho},u)\,d\vec{\rho} \quad \text{for each } u>0$$

$\mu_u(B)$ is a non-negative completely additive set function on the Lebesgue measurable sets in $\mathbb{R}^3$, moreover $\mu_u(\mathbb{R}^3) \le a$. We see that

$1-h(\vec{y},u) = \int \frac{1}{|\vec{\rho}-\vec{y}|}\mu_u(d\vec{\rho})$, since the set functions $\mu_u(\cdot)$ are uniformly bounded in u we can select, by the Helly selection principle a sequence $u_n \nearrow \infty$ where $\mu_{u_n}(B) \to \mu(B)$ where $\mu(B)$ is again a non-negative completely additive set function and $\mu(\mathbb{R}^3) \leq a$. So for $\vec{y} \notin \Omega$ we have

$$V(\vec{y}) = \lim_{u_n \to \infty} (1 - h(\vec{y}, u_n)) = \int \frac{1}{|\vec{\rho} - \vec{y}|} \mu(d\vec{\rho}) .$$

Aside: In $\mathbb{R}^d$ $d \geqslant 3$. $T_{S(0,\lambda)}(0, \vec{r}(\cdot))$. What is

$$P\left\{ \overline{\lim_{\lambda \to 0}} \frac{T_{S_\lambda}(0, \vec{r}(\cdot))}{\lambda^2 \log\log(\frac{1}{\lambda})} = \frac{2}{\rho_d^{\,2}} \right\} = 1$$

where ρ_d is the 1^{st} positive zero of $J_\gamma(x)$, $\gamma = \frac{d}{2} - 2$.