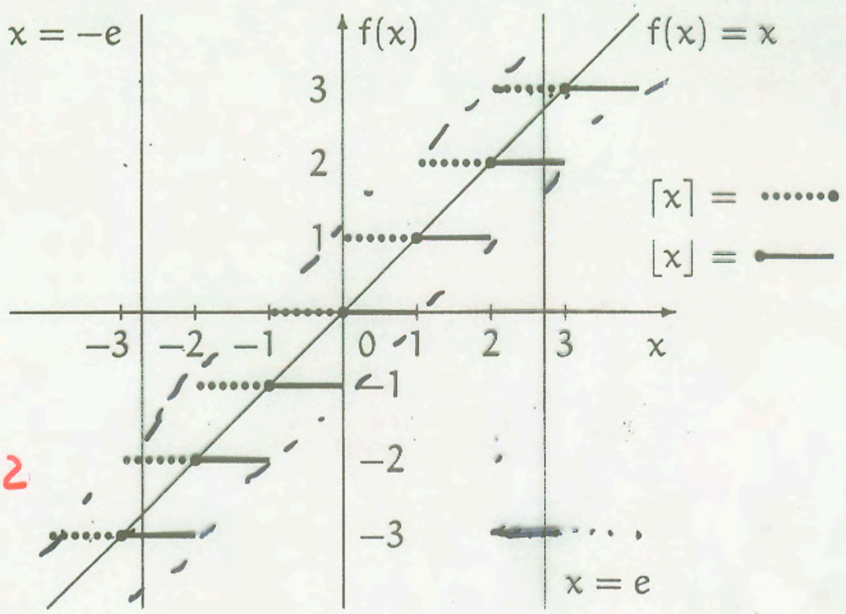
Integer Functions



$\lfloor -e \rfloor = -3$

$\lceil -e \rceil = -2$

$\lfloor e \rfloor = 2$

$\lceil e \rceil = 3$

Floor: $\lfloor x \rfloor$

Ceiling: $\lceil x \rceil$ graphs

Many calculators & languages have Int(x) function (rounds towards zero), and has the property $\mathrm{Int}(-x) = -\mathrm{Int}(x)$.

Facts: $\lfloor x \rfloor \le x \le \lceil x \rceil$

equality iff x is an integer

$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}]$

By shifting the graphs we see that

$$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$$

And $\lfloor -x \rfloor = -\lceil x \rceil$; $\lceil -x \rceil = -\lfloor x \rfloor$, thus one can express one in terms of the other.

Some useful identities (for proofs, etc.)

$$\lfloor x \rfloor = n \iff n \le x < n+1$$
$$\lfloor x \rfloor = n \iff x-1 < n \le x$$
$$\lceil x \rceil = n \iff n-1 < x \le n$$
$$\lceil x \rceil = n \iff x \le n < x+1$$

we assume $n \in \mathbb{Z}$, $x \in \mathbb{R}$

Clearly we may subtract out integers:

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n$$

$$\lfloor 2 \cdot \tfrac{1}{2} \rfloor = 1 \neq \lfloor 2 \rfloor \cdot \lfloor \tfrac{1}{2} \rfloor = 0$$

Often floor/ceiling are redundant

$$x < n \iff \lfloor x \rfloor < n$$
$$n < x \iff n < \lceil x \rceil$$
$$x \le n \iff \lceil x \rceil \le n$$
$$n \le x \iff n \le \lfloor x \rfloor$$

A related concept: fractional part of x:

$$\{x\} = x - \lfloor x \rfloor$$

"integer part of x"

If x can be written as $x = n + \theta$
$n \in \mathbb{Z}$
$0 \le \theta < 1$, then

then $n = \lfloor x \rfloor$ and $\theta = \{x\}$.

Consider $\lfloor x+y \rfloor = \lfloor \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} \rfloor$

$= \lfloor \{x\} + \{y\} \rfloor + \lfloor x \rfloor + \lfloor y \rfloor$ integer

$0 \le \{x\} + \{y\} < 2$ is either 0 or 1.

Floor and Ceiling Applications

What is $\lceil \lg 35 \rceil$: $2^5 < 35 \le 2^6$

$5 < \lg 35 \le 6$

so $\lceil \lg 35 \rceil = 6$.

Note: $35 = (100011)_2$ is 6-bits long

so is $\lceil \lg n \rceil$ the length of n in bits?

No: $32 = (100000)_2$ and $\lceil \lg 32 \rceil = 5$.

each number n has m bits when

$$2^{m-1} \le n < 2^m$$

so $m-1 = \lfloor \lg n \rfloor \iff m = \lfloor \lg n \rfloor + 1$

also $m = \lceil \lg (n+1) \rceil$ and this holds for $n = 0$ as well.

What is $\lceil \lfloor x \rfloor \rceil$? Since $\lfloor x \rfloor \in \mathbb{Z}$, $= \lfloor x \rfloor$.
Clearly nesting floor/ceiling expressions has only the inner one being important.

Prove or disprove: $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$, $x \geq 0$

Clearly holds when $x = \lfloor x \rfloor$

Trying examples π, e, ϕ, does not give counterexample.

Lets ~~try~~ to prove it.

Don't try $x = \lfloor x \rfloor + \theta$!

Define $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor \iff m \leq \sqrt{\lfloor x \rfloor} < m+1$ (square)

$m^2 \leq \lfloor x \rfloor < (m+1)^2 \iff m^2 \leq x < (m+1)^2$

$(n \leq x \Leftrightarrow n \leq \lfloor x \rfloor)$ $\iff m \leq \sqrt{x} < m+1$ (square root)

$(x < n \Leftrightarrow \lfloor x \rfloor < n)$ $\iff m = \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$

Can also prove $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$, $x \geq 0$.

Doesn't really depend on square root function much. Let $f(x)$ be continuous, monotonically increasing and

$$f(x) \in \mathbb{Z} \implies x \in \mathbb{Z}, \text{ then}$$

$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor$ and $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$

Proof of $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$. If $x = \lceil x \rceil$ there is nothing to prove. So $x < \lceil x \rceil \Longrightarrow f(x) < f(\lceil x \rceil)$ as f is increasing. Since ceiling is a non-decreasing function $\lceil f(x) \rceil \leq \lceil f(\lceil x \rceil) \rceil$. If $<$ holds, $\exists y \ni x \leq y < \lceil x \rceil$ and $f(y) = \lceil f(x) \rceil$ since f is continuous. But $\lceil f(x) \rceil \in \mathbb{Z}$ so $f(y) \in \mathbb{Z} \Rightarrow y \in \mathbb{Z}$, which is a contradiction so $\not<$ and $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$. $\square$

Important special case:

$$\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor ; \quad \left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil$$

$m, n \in \mathbb{Z}$, $n > 0$ $\quad \lfloor \lfloor \lfloor x/10 \rfloor /10 \rfloor /10 \rfloor = \lfloor x/1000 \rfloor$

What about mixed-mode?

$$\lceil \sqrt{\lfloor x \rfloor} \rceil \stackrel{?}{=} \lceil \sqrt{x} \rceil, \quad x \geq 0$$

Fails with $\phi = x$, so false. This begs the general question different math problem levels.

1: Given x explicitly, prove $P(x)$: "prove $\lfloor \pi \rfloor = 3$"

2: Given $\mathbb{X}$ prove $P(x)$ $\forall x \in \mathbb{X}$: "prove $\lfloor x \rfloor \leq x$ $\forall x \in \mathbb{R}$"

3: Given $\mathbb{X}$ and $P(x)$, prove or disprove $P(x)$ $\forall x \in \mathbb{X}$.

"Prove or disprove $\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil$ $\forall x \in \mathbb{R} \geq 0$."

Must first try to find a counter example, or lacking that, try to prove it. Very similar to "real math."

4: Given $\mathbb{X}$ and $P(x)$, find a necessary and sufficient condition $Q(x)$ that makes $P(x)$ true.
"Find a necessary and sufficient condition so $\underbrace{\text{that } \lfloor x \rfloor \geq \lceil x \rceil}$." Must find $Q(x) \ni Q(x) \Longleftrightarrow P(x)$.

$$x \in \mathbb{Z} = Q(x)$$

After you find $Q(x)$, must prove $\Longleftrightarrow$.

5. Given $\mathbb{X}$ find an interesting property $P(x)$.
This is pure research

What are necessary and sufficient conditions so that $\lceil \sqrt{\lfloor x \rfloor} \rceil \overset{?}{=} \lceil \sqrt{x} \rceil$. We saw this failed for $\phi = 1.618$ + can see it fails for $9 < x < 10$. The bad cases occure when $m^2 < x < m^2 + 1$. Thus the condition is : $x \in \mathbb{Z}$ or $\sqrt{\lfloor x \rfloor} \notin \mathbb{Z}$.

Intervals

$\alpha \leq x \leq \beta$:	$[\alpha \,..\, \beta]$	closed
$\alpha < x \leq \beta$:	$(\alpha \,..\, \beta]$	half-open
$\alpha \leq x < \beta$:	$[\alpha \,..\, \beta)$	half-open
$\alpha < x < \beta$:	$(\alpha \,..\, \beta)$	open

How many integers are in each of these intervals?
Start with half-open, which are additive:

$$[\alpha \,..\, \beta) \cup [\beta \,..\, \gamma) = [\alpha \,..\, \gamma)$$

If $\alpha, \beta \in \mathbb{Z}$ then $[\alpha, \beta)$ contains $\alpha, \alpha+1, \ldots \beta-1$ when $\alpha \leq \beta$: $\beta - \alpha$

Similarly $(\alpha..\beta]$ contains $\beta-\alpha$ integers. What if $\alpha,\beta\in\mathbb{R}$:

$$\alpha\le n<\beta \iff \lceil\alpha\rceil\le n<\lceil\beta\rceil$$
$$\alpha< n\le\beta \iff \lfloor\alpha\rfloor< n\le\lfloor\beta\rfloor$$

Thus we convert to integer intervals:

$[\alpha..\beta)$ contains $\lceil\beta\rceil-\lceil\alpha\rceil$ integers

$(\alpha..\beta]$ contains $\lfloor\beta\rfloor-\lfloor\alpha\rfloor$ integers

Also, can show that $[\alpha..\beta]$ contains $\lfloor\beta\rfloor-\lceil\alpha\rceil+1$ integers, and $(\alpha..\beta)$ contains $\lceil\beta\rceil-\lfloor\alpha\rfloor-1$ integers. Here we must also impose $\alpha\neq\beta$.

Summary			
	$[\alpha..\beta]$	$\lfloor\beta\rfloor-\lceil\alpha\rceil+1$	$\alpha\le\beta$
	$[\alpha..\beta)$	$\lceil\beta\rceil-\lceil\alpha\rceil$	$\alpha\le\beta$
	$(\alpha..\beta]$	$\lfloor\beta\rfloor-\lfloor\alpha\rfloor$	$\alpha\le\beta$
	$(\alpha..\beta)$	$\lceil\beta\rceil-\lfloor\alpha\rfloor-1$	$\alpha<\beta$

Consider a roulette wheel with 1000 slots, that pays off when the result, n, is divisible by the floor of it's cube root:

$$\lfloor\sqrt[3]{n}\rfloor \setminus n \qquad \text{"divides"}$$

In this case the house pay \$5 on a \$1 bet, otherwise we loose our \$1.

What are the "odds" of this game?

Let's compute the average winnings: in 1 to 1000 count # of winners $= W$; $L = 1000 - W$ is the # of losers. So the average is:

$$\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}$$

If $6W - 1000 > 0$ we "beat the house" $\Rightarrow$

$W > \frac{1000}{6} = 166\frac{2}{3}$, so $W \geq 167$ "beats the house"

$\lfloor \sqrt[3]{n} \rfloor$	Range of n's	#
1	1 to $7 = 2^3 - 1$	7
2	8 to $26 = 3^3 - 1$	10
3	3^3 to $4^3 - 1$	13
$\vdots$	$\vdots$	
j	j^3 to $(j+1)^3 - 1$	?

So the winners are those n's in each group divisibly by $\lfloor \sqrt[3]{n} \rfloor$.

$$W = \sum_{n=1}^{1000} [n \text{ is a winner}]$$

$$= \sum_{1 \leq n \leq 1000} [\lfloor \sqrt[3]{n} \rfloor \backslash n] = \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][k \backslash n][1 \leq n \leq 1000]$$

$$= \sum_{k,m,n} [k^3 \le n < (k+1)^3][n = km][1 \le n \le 1000]$$

1000

$$= 1 + \sum_{k,m} [k^3 \le km < (k+1)^3][1 \le k < \sqrt[3]{1000}]$$

$$= 1 + \sum_{k,m} [m \in [k^2 \,..\, (k+1)^3/k)][1 \le k < 10]$$

$\Updownarrow$ # of ints. in interval

$$= 1 + \sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil \right)$$

$$= 1 + \sum_{1 \le k < 10} (3k+4) = 1 + 3 \cdot \frac{9 \cdot 10}{2} + 4 \cdot 9$$

$$= 1 + 3 \cdot 45 + 36$$

$$= 1 + 135 + 36 = 172$$

So $\dfrac{6 \cdot 172 \cancel{- 1000}}{1000} = +0.032$, so this favors us.

More generally: how many integers, n, $1 \le n \le N$ satisfy $\lfloor \sqrt[3]{n} \rfloor \backslash n$? Must take this sum up to $K = \lfloor \sqrt[3]{N} \rfloor$.

$$W = \sum_{1 \le k < K} (3k+4) + \sum_m [K^3 \le Km \le N]$$

$$= \frac{3}{2} K(K-1) + 4(K-1) + \sum_m [m \in [K^2 \,..\, N/K]] =$$

$$= \tfrac{3}{2}K^2 + \tfrac{5}{2}K - 4 + \lfloor N/K \rfloor - \lceil K^2 \rceil + 1$$

$$= \lfloor N/K \rfloor + \tfrac{1}{2}K^2 + \tfrac{5}{2}K - 3 \;, \quad K = \boxed{\lfloor \sqrt[3]{N} \rfloor}$$

So $w = \frac{3}{2} N^{2/3} + \mathcal{O}(N^{1/3})$ as $\lfloor N/K \rfloor \simeq K^2$

↑ get smaller with N in relative terms

N	$\frac{3}{2}N^{2/3}$	w	% error	
10^3	150.0	172	12.791	
				) ½
10^4	696.2	746	6.670	
				) ½
10^5	3231.7	3343	3.331	
				) ½ approx.
10^6	15000.0	15247	1.620	
				) ½ why?
10^7	69623.8	70158	0.761	
				) ½
10^8	323165.2	324322	0.357	
				) ½
10^9	1500000.0	1502497	0.166	

Let $\alpha \in \mathbb{R}$ and define its "spectrum" as

$$\mathrm{Spec}(\alpha) = \{ \lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \dots \}$$

this is a multiset, as it can have repeated values. Can prove that $\alpha \neq \beta \Rightarrow \mathrm{Spec}(\alpha) \neq \mathrm{Spec}(\beta)$. Assume $\alpha < \beta$, then $\exists\, m \in \mathbb{Z}^+$ so that $m(\beta - \alpha) \geq 1 \Rightarrow m\beta - m\alpha \geq 1 \Rightarrow \lfloor m\beta \rfloor > \lfloor m\alpha \rfloor$. Thus $\mathrm{Spec}(\beta)$ has less than m elements $\leq \lfloor m\alpha \rfloor$, while $\mathrm{Spec}(\alpha)$ has at least m. Consider

$$\mathrm{Spec}(\sqrt{2}) = \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, \dots\}$$

$$\mathrm{Spec}(2+\sqrt{2}) = \{3, 6, 10, 13, 17, 20, 23, 27, 30, 34, 37, 40, 44, \dots\}$$

$\mathrm{Spec}(\sqrt{2})_n + 2n = \mathrm{Spec}(2+\sqrt{2})_n$

Also it seems that $j \in \mathbb{Z}^+$ is either in $\mathrm{Spec}(\sqrt{2})$ or $\mathrm{Spec}(2+\sqrt{2})$, but not both.

This is true and we say that $\mathrm{Spec}(\sqrt{2})$ and $\mathrm{Spec}(2+\sqrt{2})$ form a partition of $\mathbb{Z}^+$.

Proof: We will count $\# \mathrm{Spec}(\sqrt{2}) \le n$ and

$\# \mathrm{Spec}(2+\sqrt{2}) \le n$

If these two always sum up to n, then they are a partition.

Let $\alpha \in \mathbb{R}$ be positive

$$N(\alpha, n) = \# \mathrm{Spec}(\alpha) \le n = \sum_{0<k} [\lfloor k\alpha \rfloor \le n]$$

$$a \le b \iff a < b+1$$

$$= \sum_{0<k} [\lfloor k\alpha \rfloor < n+1]$$

$$= \sum_{0<k} [k\alpha < n+1]$$

$$= \sum_{k} \left[k \in \left(0 \,..\, \frac{(n+1)}{\alpha}\right)\right]$$

$$= \left\lceil \frac{n+1}{\alpha} \right\rceil - \lfloor 0 \rfloor - 1 = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1$$

What is $N(\sqrt{2}, n) + N(2+\sqrt{2}, n)$?

$$\left\lceil \frac{n+1}{\sqrt{2}} \right\rceil - 1 + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 1 = \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor$$

$$= \frac{n+1}{\sqrt2} - \left\{\frac{n+1}{\sqrt2}\right\} + \frac{n+1}{2+\sqrt2} - \left\{\frac{n+1}{2+\sqrt2}\right\}$$

Note $\frac{1}{\sqrt2} + \frac{1}{2+\sqrt2} = \frac{2+\sqrt2+\sqrt2}{\sqrt2(2+\sqrt2)} = \frac{2\sqrt2+2}{2\sqrt2+2} = 1$

$$= (n+1)\left(\frac{1}{\sqrt2} + \frac{1}{2+\sqrt2}\right) - \left[\left\{\frac{n+1}{\sqrt2}\right\} + \left\{\frac{n+1}{2+\sqrt2}\right\}\right]$$

$$= n+1 - \left[\left\{\frac{n+1}{\sqrt2}\right\} + \left\{\frac{n+1}{2+\sqrt2}\right\}\right]$$

but since $\frac{n+1}{\sqrt2} + \frac{n+1}{2+\sqrt2} = n+1$, and these are never integers for any n, the fractional parts must add up to 1, and so

$$= n+1-1 = n.$$

Floor / Ceiling Recurrences

Consider $K_0 = 1;$

$$K_{n+1} = 1 + \min\left(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}\right), n \geq 0$$

$K_1 = 1 + \min(2K_0, 3K_0) = 1+2 = 3$. And this begins $1, 3, 3, 4, 7, 7, 7, 9, 9, 10, 13, \ldots$

these are called the Knuth numbers, and it seems that $K_n \geq n \;\forall n \geq 0$.

Attempted proof by induction:

$n=0$ (basis) $K_0 = 1 \geq 0$ ✓

Assume $K_j \geq j \quad \forall j \leq n$ and consider

$$K_{n+1} = 1 + \min\left(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}\right)$$

$$2K_{\lfloor n/2 \rfloor} \geq 2\left\lfloor \tfrac{n}{2} \right\rfloor \quad \text{and} \quad 3K_{\lfloor n/3 \rfloor} \geq 3\left\lfloor \tfrac{n}{3} \right\rfloor$$

but $2\lfloor \frac{n}{2} \rfloor$ can be $n-1$ and $3\lfloor \frac{n}{3} \rfloor$ can be $n-2$ so we have only $K_{n+1} \geq 1 + (n-2) = n-1$ not $n+1$!

Recurrences involving floor/ceiling often arise in CS because of "divide and conquer" algorithmic analysis. In sorting we often recursively divide a list into 2 pieces of $\lfloor \frac{n}{2} \rfloor$ + $\lceil \frac{n}{2} \rceil$, note: $\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n$. With merging, and $n-1$ comparisons, we perform $f(n)$ comparisons where

$$f(1) = 0;$$
$$f(n) = f(\lceil n/2 \rceil) + f(\lfloor n/2 \rfloor) + n - 1, \quad n > 1$$

Can also rewrite Josephus as

$$J(1) = 1;$$
$$J(n) = 2J(\lfloor n/2 \rfloor) - (-1)^n, \quad n > 1$$

Consider a variant on Josephus: every 3rd person is eliminated:

$$J_3(n) = \left\lceil \tfrac{3}{2} J_3\left(\left\lfloor \tfrac{2}{3} n \right\rfloor\right) + a_n \right\rceil \bmod n+1$$

$$a_n = \begin{cases} -2 & n \equiv 0 \pmod 3 \\ +1 & \text{"} \quad 1 \quad \text{"} \\ -\frac{1}{2} & \text{"} \quad 2 \quad \text{"} \end{cases}$$

This seems pretty hard. Why not re-number as follows (example with 10)

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

Point: in this procedure the survivor is renumbered $3n$. So if we can derive the original number of $3n$, we solve the problem.

$1 \to n+1$, $2 \to n+2$, $\cancel{3}$, $4 \to n+3$, $5 \to n+4$, $\cancel{6}$, … $3k+1 \to n+2k+1$, $3k+2 \to n+2k+2$, $\cancel{3k+3}$, …

If $N > n$, N must have had a previous number, lets compute it:

$$N = \begin{cases} n + 2k + 1 \\ \quad\text{or} \\ n + 2k + 2 \end{cases} \Longrightarrow k = \lfloor (N - n - 1)/2 \rfloor$$

and the previous number was $3k+1$ or $3k+2$, or

$$3k + (N - n - 2k) = k + N - n.$$

To compute $J_3(n)$ we can do so as follows:

$$\begin{aligned}
&N := 3n; \\
&\text{while } N > n \text{ do } N := \overset{(k)}{\left\lfloor \frac{N-n-1}{2} \right\rfloor} + N - n; \\
&J_3(n) := N \quad \text{, this counts down.}
\end{aligned}$$

Let $D = 3n + 1 - N$ replace N, so

$$\begin{aligned}
D := 3n + 1 - \left(\left\lfloor \frac{(3n+1-D) - n - 1}{2} \right\rfloor + (3n+1-D) - n \right) \\
= n + D - \left\lfloor \frac{2n - D}{2} \right\rfloor = D - \left\lfloor \frac{-D}{2} \right\rfloor \\
= D + \left\lceil \frac{D}{2} \right\rceil = \left\lceil \tfrac{3}{2} D \right\rceil
\end{aligned}$$

Algorithm becomes

$$\begin{aligned}
&D := 1; \\
&\text{while } D \le 2n \text{ do } D := \left\lceil \tfrac{3}{2} D \right\rceil; \\
&J_3(n) := 3n + 1 - D.
\end{aligned}$$

this counts up, not down.

Can we generalise this to $J_q(n)$?

$$D := 1;$$
$$\text{while } D \le (q-1)n \text{ do } D := \left\lceil \frac{q}{q-1} D \right\rceil;$$
$$J_q(n) := qn + 1 - D$$

$q = 2$ is our "old friend": $\frac{q}{q-1} = 2$, so we grow D to 2^{m+1} with $n = 2^m + \ell$

$$\begin{aligned} J_2(n) &= 2n + 1 - D \\ &= 2(2^m + \ell) + 1 - 2^{m+1} \\ &= 2^{m+1} + 2\ell + 1 - 2^{m+1} = 2\ell + 1 \checkmark \end{aligned}$$

Define $D_0^{(q)} = 1$; $D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil$ $n > 0$.

then $J_q(n) = qn + 1 - D_k^{(q)}$ where k is the smallest index such that $D_k^{(q)} > (q-1)n$.

'MOD': The Binary Operation

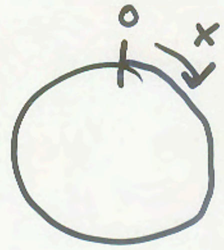
Quotient / Remainder (given $n \in \mathbb{Z}$, $m \in \mathbb{Z}^+$)

$$n = m \underbrace{\left\lfloor \frac{n}{m} \right\rfloor}_{\text{quotient}} + \underbrace{n \bmod m}_{\text{remainder}}$$

$$x \bmod y = x - y\left\lfloor \frac{x}{y} \right\rfloor, \quad y \neq 0$$

for real x, y too.

Interpretation:



~ radius is $\frac{y}{2\pi}$ so

circumference is y

$x \bmod y$ is final location

$\left\lfloor \frac{x}{y} \right\rfloor$ is number of full circles

To see impact of negative numbers consider:

$$5 \bmod 3 = 5 - 3\underset{(1)}{\lfloor 5/3 \rfloor} = 2; \quad 5 \bmod -3 = 5 - (-3)\overset{(-2)}{\lfloor 5/-3 \rfloor} = -1$$

$$-5 \bmod 3 = -5 - 3\underset{(-2)}{\lfloor -5/3 \rfloor} = 1; \quad -5 \bmod -3 = -5 - (-3)\underset{(1)}{\lfloor -5/-3 \rfloor} = -2$$

these are called the modulus

Mostly modulus is positive, but

$$0 \leq x \bmod y < y \quad \text{for } y > 0$$

$$0 \geq x \bmod y > y \quad \text{for } y < 0$$

With $y = 0$ we can define $x \bmod 0 = x$

Recall $x = \lfloor x \rfloor + \underbrace{\{x\}} \rightarrow x \bmod 1$

Note: mod is defined via "floor" what about

$$x \text{ "mumble" } y = y\lceil x/y \rceil - x$$

this is the "distance left to zero."

Some properties:

Distributive $\quad c(x \bmod y) = (cx) \bmod (cy)$

$$\forall\, c, x, y \in \mathbb{R}$$

$$c(x \bmod y) = c(x - y\lfloor x/y \rfloor)$$
$$= cx - cy\lfloor cx/cy \rfloor = (cx) \bmod (cy)$$

An application of mod: partition n things into m groups as "evenly" as possible.

Example: n lines of text and we want to arrange them in m columns. Should be in decreasing order of length, and no two columns should differ in length by more then one line. We wish to distribute column-wise. We decide how many in the 1st column, then the second, etc.

As an example, consider $n = 37$ (lines) and $m = 5$ (columns)

Two possible arrangements are below. The one on the right is preferred, as column length differ only by one.

8	8	8	8	5
line 1	line 9	line 17	line 25	line 33
line 2	line 10	line 18	line 26	line 34
line 3	line 11	line 19	line 27	line 35
line 4	line 12	line 20	line 28	line 36
line 5	line 13	line 21	line 29	line 37
line 6	line 14	line 22	line 30	
line 7	line 15	line 23	line 31	
line 8	line 16	line 24	line 32	

8	8	7	7	7
line 1	line 9	line 17	line 24	line 31
line 2	line 10	line 18	line 25	line 32
line 3	line 11	line 19	line 26	line 33
line 4	line 12	line 20	line 27	line 34
line 5	line 13	line 21	line 28	line 35
line 6	line 14	line 22	line 29	line 36
line 7	line 15	line 23	line 30	line 37
line 8	line 16			

Clearly the long columns contain $\lceil \frac{n}{m} \rceil$ lines and the short columns $\lfloor \frac{n}{m} \rfloor$ lines. There will be $n \bmod m$ long columns and n "mumble" m short columns.

Generalize to "things" and "groups": The first group will contain $\lceil \frac{n}{m} \rceil$ things, so to distribute n things in m groups we iterate

$n' = n$
$m' = m$
for $i = 1$ to m (until $m' = 1$ / $n = 0$)
put $\lceil \frac{n'}{m'} \rceil$ things in a group
$n' = n' - \lceil \frac{n'}{m'} \rceil$
$m' = m' - 1$
do

(This is iterative, but can be defined recursively)

Example : $n = 314$, $m = 6$

$314, 6 : \left\lceil \frac{314}{6} \right\rceil = 53 \to 261, 5 : \left\lceil \frac{261}{5} \right\rceil = 53 \to$
$208, 4 : \left\lceil \frac{208}{4} \right\rceil = 52 \to 156, 3 : \left\lceil \frac{156}{3} \right\rceil = 52 \to 104, 2 :$
$\left\lceil \frac{104}{2} \right\rceil = 52 \to 52, 1 : \left\lceil \frac{52}{1} \right\rceil = 52 : \; 53, 53, 52, 52, 52, 52$

How many things are in the $k^{\underline{th}}$ group?

$$? = \begin{cases} \lceil n/m \rceil & \text{when } k \leq n \bmod m \\ \lfloor n/m \rfloor & \text{when } k > n \bmod m \end{cases}$$

$$\left\lceil \frac{n-k+1}{m} \right\rceil = \left\lceil \frac{qm + r - k + 1}{m} \right\rceil = q + \left\lceil \frac{r-k+1}{m} \right\rceil$$

and $\left\lceil \frac{r-k+1}{m} \right\rceil = [k \leq r]$ with $1 \leq k \leq m, 0 \leq r < m$.

Since the sum of each group must equal the total we also have

$$n = \sum_{k=1}^{m} \left\lceil \frac{n-k+1}{m} \right\rceil = \left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{n-1}{m} \right\rceil + \cdots + \left\lceil \frac{n-m+1}{m} \right\rceil$$

with $m = 2 \implies n = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$

What if we want the groups to be presented in nondecreasing order (smaller groups first):

$$n = \sum_{k=1}^{m} \left\lfloor \frac{n+k-1}{m} \right\rfloor = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \cdots + \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

If we replace n by $\lfloor mx \rfloor$ in the previous, we get:

$$\lfloor mx \rfloor = \sum_{k=1}^{m} \left\lfloor x + \frac{k-1}{m} \right\rfloor = \underbrace{\lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + \cdots + \left\lfloor x + \frac{m-1}{m} \right\rfloor}_{m \text{ terms}}$$

This is remarkable. To see why assume $\lfloor x \rfloor \approx x - \frac{1}{2}$ "on average"

$$mx - \frac{1}{2} : \text{lhs} \qquad \left(x - \frac{1}{2}\right) + \left(x - \frac{1}{2} + \frac{1}{m}\right) + \left(x - \frac{1}{2} + \frac{2}{m}\right) + \cdots$$

$$+ \left(x - \frac{1}{2} + \frac{m-1}{m}\right) : \text{rhs}$$

$$= mx - \frac{m}{2} + \frac{1}{m}\frac{m(m-1)}{2}$$

$$= mx - \frac{m}{2} + \frac{m}{2} - \frac{1}{2} = mx - \frac{1}{2} \quad !$$

Floor / Ceiling Sums

Consider $\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor$, let $m = \lfloor \sqrt{k} \rfloor$

$$= \sum_{k, m \geq 0} m [k < n][m = \lfloor \sqrt{k} \rfloor]$$

$$= \sum_{k, m \geq 0} m [k < n][m \le \sqrt{k} < m+1] = \sum_{k, m \geq 0} m [k < n][m^2 \le k < (m+1)^2]$$

$$= \sum_{k, m \geq 0} m [m^2 \le k < (m+1)^2 \le n] + \sum_{k, m \geq 0} m [m^2 \le k < n < (m+1)^2]$$

The boundary conditions are tricky, first assume $n = q^2$ is a perfect square.

the second sum is zero:

$$\sum_{k,m\ge 0} m\left[m^2 \le k < (m+1)^2 \le a^2\right] = \sum_{m\ge 0} m\left((m+1)^2 - m^2\right)\left[m+1 \le a\right]$$

$$= \sum_{m\ge 0} m(2m+1)\left[m<a\right] = \sum_{m\ge 0}\left(2m^{\underline{2}} + 3m^{\underline{1}}\right)\left[m<a\right]$$

$$= \sum_0^a \left(2m^{\underline{2}} + 3m^{\underline{1}}\right)\delta m = \frac{2}{3}a(a-1)(a-2) + \frac{3}{2}a(a-1)$$

$$= \frac{1}{6}(4a+1)a(a-1)$$

When $a = \lfloor\sqrt{n}\rfloor$, we must consider the second sum, which are for $a^2 \le k < n$. Each term equals a, and there are $(n-a^2)$ of them, so

$$\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a\,, \quad a = \lfloor\sqrt{n}\rfloor$$

(adding in $(n-a^2).a$)

Consider replacing $\lfloor x\rfloor$ by $\sum_j [1\le j\le x]$, $n=a^2$:

$$\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = \sum_{j,k}\left[1\le j\le\sqrt{k}\right]\left[0\le k<a^2\right]$$

$$= \sum_{1\le j<a}\ \sum_k \left[j^2 \le k < a^2\right]$$

$$= \sum_{1\le j<a}\left(a^2 - j^2\right) = a^3 - \frac{1}{3}a\left(a+\frac{1}{2}\right)(a+1)$$

Theorem: (Bohl, Sierpinski, Weyl) If α is irrational then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0\le k<n} f(\{k\alpha\}) = \int_0^1 f(x)\,dx$$

with all bounded and continuous (almost everywhere) $f(x)$. If $f(x) = x$ then $\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{0\le k<n} \{k\alpha\}$$

Consider $f_v(x) = [0 \le x < v]$, then we wish to see how close

$$\sum_{0\le k<n} [\{k\alpha\} < v] \text{ is to } nv.$$

Define $s(\alpha, n, v) = \sum_{0\le k<n} ([\{k\alpha\} < v] - v)$ and

$$D(\alpha, n) = \sup_{0\le v\le 1} |s(\alpha, n, v)|$$

we wish to show $D(\alpha, n)$ is not too large compared to n, by showing $|s(\alpha, n, v)|$ is reasonably small when α is irrational.

$$\sum_{0\le k<n} ([\{k\alpha\} < v] - v) = \sum_{0\le k<n} (\lfloor k\alpha \rfloor - \lfloor k\alpha - v \rfloor - v) =$$

$$= -nr + \sum_{0 \le k < n} \sum_j [k\alpha - r < j \le k\alpha]$$

$$= -nr + \sum_{0 \le j < \lceil n\alpha \rceil} \sum_{k < n} [j\alpha^{-1} \le k < (j+r)\alpha^{-1}]$$

Can assume $0 < \alpha < 1$, and define:

$$a = \lfloor \alpha^{-1} \rfloor, \quad \alpha^{-1} = \lfloor \alpha^{-1} \rfloor + \alpha'$$

$$b = \lceil r\alpha^{-1} \rceil, \quad r\alpha^{-1} = \lceil r\alpha^{-1} \rceil - r'$$

so $\alpha' = \{\alpha^{-1}\}$ & r' is the "mumble" fractional part of $r\alpha^{-1}$.

Remove the boundary condition $k < n$:

$$\sum_k [k \in [j\alpha^{-1}, (j+r)\alpha^{-1})] =$$

$$\lceil (j+r)(a+\alpha') \rceil - \lceil j(a+\alpha') \rceil =$$

$$\lceil ja + j\alpha' + r\alpha^{-1} \rceil - \lceil ja + j\alpha' \rceil =$$

integers cancel

$$= \lceil j\alpha' + b - r' \rceil - \lceil j\alpha' \rceil = b + \lceil j\alpha' - r' \rceil - \lceil j\alpha' \rceil$$

So $s(\alpha, n, r) = -nr + \lceil n\alpha \rceil b +$

$$\sum_{0 \le j < \lceil n\alpha \rceil} (\lceil j\alpha' - r' \rceil - \lceil j\alpha' \rceil) - S$$

correction for $k \ge n$ not excluded

$j\alpha'$ will be an integer only when $j=0$ since α' is also irrational. Similarly $j\alpha'-r'$ will be an integer for at most one j. So we can change ceilings into floors

$$s(\alpha,n,r) = -nr + \lceil n\alpha\rceil b - \sum_{0\le j<\lceil n\alpha\rceil}\left(\lfloor j\alpha'\rfloor - \lfloor j\alpha'-r'\rfloor\right) - S + \{0 \text{ or } 1\}$$

Note $s(\alpha', \lceil n\alpha\rceil, r') = \sum_{0\le j<\lceil n\alpha\rceil}\left(\lfloor j\alpha'\rfloor - \lfloor j\alpha'-r'\rfloor - r'\right)$

can be related as:

$$s(\alpha,n,r) = -nr + \lceil n\alpha\rceil b - \lceil n\alpha\rceil r' - s(\alpha',\lceil n\alpha\rceil, r') - S + \varepsilon + \{0 \text{ or } 1\}$$

Here $0<\varepsilon<r\alpha'$, and $0<S<\lceil r\alpha^{-1}\rceil$. We also remove the term for $j=\lceil n\alpha\rceil-1=\lfloor n\alpha\rfloor$ since it contributes either r' or $r'-1$. Max over absolute values and r to get

$$D(\alpha,n) \le D(\alpha', \lfloor\alpha n\rfloor) + \alpha^{-1}+2$$

this implies (by more advanced methods) that

$$D(\alpha,n) = o(n).$$

Last sum:

$$\sum_{0\le k<m}\left\lfloor\frac{nk+x}{m}\right\rfloor, \quad m\in\mathbb{Z}^+,\ n\in\mathbb{Z}$$

With $n=1$ we get an old friend:

$$\lfloor x \rfloor = \left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x+1}{m} \right\rfloor + \dots + \left\lfloor \frac{x+m-1}{m} \right\rfloor$$

which is what we saw before but with $\frac{x}{m}$ instead of x.

With $n=0$

$$\sum_{0 \le k < m} \left\lfloor \frac{x}{m} \right\rfloor = m \left\lfloor \frac{x}{m} \right\rfloor.$$ What about for small m?

$m=2$: $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+n}{2} \right\rfloor = 2\left\lfloor \frac{x}{2} \right\rfloor + \frac{n}{2}$ $\quad$ n even

$= \lfloor x \rfloor + \frac{n-1}{2}$ $\quad$ n odd

$m=3$: Cases $n \bmod 3 = 0, 1, 2$.

0: $\frac{n}{3}$, $\frac{2n}{3}$ are integers

$$\left\lfloor \frac{x}{3} \right\rfloor + \left(\left\lfloor \frac{x}{3} \right\rfloor + \frac{n}{3} \right) + \left(\left\lfloor \frac{x}{3} \right\rfloor + \frac{2n}{3} \right) = 3\left\lfloor \frac{x}{3} \right\rfloor + n$$

1: $\frac{n-1}{3}$, $\frac{2n-2}{3}$ are integers

$$\left\lfloor \frac{x}{3} \right\rfloor + \left(\left\lfloor \frac{x+1}{3} \right\rfloor + \frac{n-1}{3} \right) + \left(\left\lfloor \frac{x+2}{3} \right\rfloor + \frac{2n-2}{3} \right) = \lfloor x \rfloor + n - 1$$

2: $\frac{n-2}{3}$, $\frac{2n-1}{3}$ are integers

$$\left\lfloor \frac{x}{3} \right\rfloor + \left(\left\lfloor \frac{x+2}{3} \right\rfloor + \frac{n-2}{3} \right) + \left(\left\lfloor \frac{x+1}{3} \right\rfloor + \frac{2n-1}{3} \right) = \lfloor x \rfloor + n - 1$$

$m=4$: $\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+n}{4} \right\rfloor + \left\lfloor \frac{x+2n}{4} \right\rfloor + \left\lfloor \frac{x+3n}{4} \right\rfloor$

$n \bmod 4 = 0$: $\frac{n}{4}$, $\frac{2n}{4}$, $\frac{3n}{4}$ are integers

$$\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x}{4} \right\rfloor + \frac{n}{4} + \left\lfloor \frac{x}{4} \right\rfloor + \frac{2n}{4} + \left\lfloor \frac{x}{4} \right\rfloor + \frac{3n}{4} = 4\left\lfloor \frac{x}{4} \right\rfloor + \frac{3n}{2}$$

1: $\frac{n-1}{4}$, $\frac{2n-2}{4}$, $\frac{3n-3}{4}$ are integers

$$\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+1}{4} \right\rfloor + \frac{n-1}{4} + \left\lfloor \frac{x+2}{4} \right\rfloor + \frac{2n-2}{4} + \left\lfloor \frac{x+3}{4} \right\rfloor + \frac{3n-3}{4}$$

$$= \lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}$$

3: $\frac{n+1}{4}$, $\frac{2n+2}{4}$, $\frac{3n+3}{4}$ are integers

$$\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x-1}{4} \right\rfloor + \frac{n+1}{4} + \left\lfloor \frac{x-2}{4} \right\rfloor + \frac{2n+2}{4} + \left\lfloor \frac{x-3}{4} \right\rfloor + \frac{3n+3}{4} =$$

$$\underbrace{\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x-1}{4} \right\rfloor + \left\lfloor \frac{x-2}{4} \right\rfloor + \left\lfloor \frac{x-3}{4} \right\rfloor} + \frac{3n}{2} + \frac{3}{2}$$

But this is shifted by 3

$$\lfloor x \rfloor - 3 \qquad , \quad \lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}.$$

2: $\frac{n-2}{4}$, $\frac{2n}{4}$, $\frac{3n-2}{4}$ are integers

$$\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+2}{4} \right\rfloor + \frac{n-2}{4} + \left\lfloor \frac{x}{4} \right\rfloor + \frac{2n}{4} + \left\lfloor \frac{x+2}{4} \right\rfloor + \frac{3n-2}{4} =$$

$$2\left(\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+2}{4} \right\rfloor\right) + \frac{3n}{2} - 1 = 2\left\lfloor \frac{x}{2} \right\rfloor + \frac{3n}{2} - 1$$

Table

m \ n mod m	0	1	2	3
1	$\lfloor x \rfloor$	—	—	—
2	$2\lfloor \frac{x}{2} \rfloor + \frac{n}{2}$	$\lfloor x \rfloor + \frac{n}{2} - \frac{1}{2}$	—	—
3	$3\lfloor \frac{x}{3} \rfloor + n$	$\lfloor x \rfloor + n - 1$	$\lfloor x \rfloor + n - 1$	—
4	$4\lfloor \frac{x}{4} \rfloor + \frac{3n}{2}$	$\lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}$	$2\lfloor \frac{x}{2} \rfloor + \frac{3n}{2} - 1$	$\lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}$

Seems to be of the form

$$a\left\lfloor \frac{x}{a} \right\rfloor + bn + c \quad , \quad b = \frac{m-1}{2} \text{ is a good guess}$$

a seems to be $\gcd(m,n)$

We have alway rewritten as $\left\lfloor \frac{x+kn}{m} \right\rfloor =$

$$\left\lfloor \frac{x + kn \bmod m}{m} \right\rfloor + \frac{kn}{m} - \frac{kn \bmod m}{m}$$

because $\frac{kn - kn \bmod m}{m}$ is an integer:

$$\begin{aligned}
& \left\lfloor \frac{x}{m} \right\rfloor + \frac{0}{m} - \frac{0 \bmod m}{m} \\
+ & \left\lfloor \frac{x + n \bmod m}{m} \right\rfloor + \frac{n}{m} - \frac{n \bmod m}{m} \\
+ & \left\lfloor \frac{x + 2n \bmod m}{m} \right\rfloor + \frac{2n}{m} - \frac{2n \bmod m}{m} \\
& \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\
+ & \left\lfloor \frac{x + (m-1)n \bmod m}{m} \right\rfloor + \frac{(m-1)n}{m} - \frac{(m-1)n \bmod m}{m}
\end{aligned}$$

b is from

$$\frac{n}{m} \sum_{i=1}^{m-1} i = \frac{n}{m} \frac{(m-1)(m)}{2} = n \cdot \underbrace{\frac{m-1}{2}}_{b.}$$

To find a & c we must consider the sequence

$0 \bmod m, \; n \bmod m, \; 2n \bmod m, \; \ldots, \; (m-1)n \bmod m$

Look at $m=12$ $n=5$ (time on a clock)

$0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7$

(full period)

$m=12$ $n=8$

$0, 8, 4, 0$ (short period)

note $\gcd(12,5)=1$ but $\gcd(12,4)=4$

So we get period of length $\frac{12}{\gcd(12,4)}$; and we will see we get $0, d, 2d, \dots, m-d$ in some order followed by $d-1$ more copies with $d=\gcd(m,n)$. Thus the 1^{st} column contains d copies of $\lfloor\frac{x}{m}\rfloor, \lfloor\frac{x+d}{m}\rfloor, \dots, \lfloor\frac{x+m-d}{m}\rfloor$ so it sums to

$$d\left(\left\lfloor\frac{x}{m}\right\rfloor+\left\lfloor\frac{x+d}{m}\right\rfloor+\left\lfloor\frac{x+2d}{m}\right\rfloor+\cdots+\left\lfloor\frac{x+m-d}{m}\right\rfloor\right)$$

$$= d\left(\left\lfloor\frac{x/d}{m/d}\right\rfloor+\left\lfloor\frac{x/d+1}{m/d}\right\rfloor+\cdots+\left\lfloor\frac{x/d+m/d-1}{m/d}\right\rfloor\right)$$

$$= d\left\lfloor\frac{x}{d}\right\rfloor \quad \text{so} \quad a = \gcd(m,n) = d$$

But what about c? We can now compute the last column: it is d copies of $\frac{0}{m}, \frac{1}{m}, \cdots \frac{m-d}{m}$

This is $\frac{0}{m} + \frac{1}{m} + \cdots + \frac{m-d}{m} =$

$$\frac{1}{2}\left(\frac{0}{m} + \frac{m-d}{m}\right) \cdot \frac{m}{d} = \frac{m-d}{2d}$$

$\uparrow$ 1st $\quad \uparrow$ last $\quad \uparrow$ # of terms

so $\quad d \cdot \frac{m-d}{2d} = \frac{m-d}{2} = -C.$

$$\sum_{0 \le k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor = d\left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)}{2}n + \frac{d-m}{2}$$

$$d = \gcd(m, n)$$

$$= d\left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)(n-1)}{2} + \frac{m-1}{2} + \frac{d-m}{2}$$

$$= d\left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)(n-1)}{2} + \frac{d-1}{2}$$

thus m and n can be interchanged and so

$$\sum_{0 \le k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor = \sum_{0 \le k < n} \left\lfloor \frac{mk+x}{n} \right\rfloor$$

$$\forall\, m, n \in \mathbb{Z}^+$$