Chapter 2: Sums

upper bd. — summand

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

lower bd. — ↑ dummy index

Σ-notation introduced by Fourier

The generalized Σ-notation will be preferred to the above, delimited, form (sometimes not!)

$$a_1 + a_2 + \cdots + a_n = \sum_{1 \le k \le n} a_k$$

which is generally $\sum_{P(k)} a_k$ ↑ a property

delimited →

$$\sum_{k=0}^{49} (2k+1)^2 = \sum_{1 \le k \le 100} k^2$$

← generalized

k odd

$$\sum_{k=1}^{\pi(N)} \frac{1}{p_k} = \sum_{p \le N} \frac{1}{p}$$

p prime

$$= 1 -$$

A slight variation

$$\sum_k a_k [P(k)]$$

$$[P(k)] = \begin{cases} 1 & \text{if } P(k) \text{ is true} \\ 0 & \text{if } P(k) \text{ is false} \end{cases}$$

$$\sum_p [p \text{ prime}][p \le N]/p \quad , \quad \text{"strongly } O\text{"}$$

Sums and Recurrences

$$S_n = \sum_{k=0}^{n} a_k \quad \text{is equivalent to}$$

$$\begin{cases} S_0 = a_0 \\ S_n = S_{n-1} + a_n \quad , \quad n > 0 \end{cases}$$

What if $a_n = \beta + \gamma n$, then

$R_0 = \alpha$;

$R_n = R_{n-1} + \beta + \gamma n \quad n \geq 1$

we have already solved this as

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

Using the repertoire method (remember)

$R_n = 1 = \alpha$, $\beta = 0$, $\gamma = 0$

$\Rightarrow A(n)\alpha = 1 \Rightarrow A(n) = 1.$

$R_n = n \quad \alpha = 0, \beta = 1, \gamma = 0 \Rightarrow B(n) = n$

$R_n = n^2 \quad \alpha = 0, \beta = -1, \gamma = 2 \Rightarrow$

$2C(n) - B(n) = 2C(n) - n = n^2$

$C(n) = \frac{n^2+n}{2}$

$$\sum_{0 \leq k \leq n} (a + bk) = a(n+1) + b\left(\frac{n^2+n}{2}\right)$$

$\alpha = \beta = a$

$\gamma = b$

Recall $T_0 = 0;$
$T_n = 2T_{n-1} + 1$ (divide by 2^n)

$S_n = T_n / 2^n$

$T_0 / 2^0 = S_0 = 0;$

$T_n / 2^n = S_n = T_{n-1} / 2^{n-1} + 1/2^n = S_{n-1} + 2^{-n}$

$$S_n = \sum_{k=1}^{n} 2^{-k} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$$
$$= 1 - 2^{-n} \quad \text{(derived later)}$$

$$T_n = 2^n \cdot (1 - 2^{-n}) = 2^n - 1$$

This is a special case of this general trick

$$a_n T_n = b_n T_{n-1} + c_n$$

we convert this to a sum, multiply both sides by s_n, a summation factor

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n$$

Choose $s_n b_n = s_{n-1} a_{n-1}$ and

write $S_n = s_n a_n T_n$

$$S_n = S_{n-1} + s_n c_n \text{ , thus}$$

$$S_n = s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k = s_1 b_1 T_0 + \sum s_k c_k$$

$$T_n = \frac{1}{s_n a_n} \left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right)$$

$$\text{let } n=1 : \quad T_1 = \frac{s_1 b_1 T_0 + s_1 c_1}{s_1 a_1} = \frac{b_1 T_0 + c_1}{a_1}$$

how do we ensure $s_n b_n = s_{n-1} a_{n-1}$?

$$s_n = s_{n-1} \left(\frac{a_{n-1}}{b_n} \right) = s_{n-2} \left(\frac{a_{n-1} a_{n-2}}{b_n b_{n-1}} \right)$$

$$\cdots = \underbrace{\frac{a_{n-1} a_{n-2} \cdots a_1 a_0}{b_n b_{n-1} \cdots b_2 b_1}}_{n \text{ terms}}$$

For Tower of Hanoi $a_n = 1$ $b_n = 2$ so

$s_n = 2^{-n}$ works

Must assure b's not zero.

Consider the following from quicksort analysis, for the average number of comparisons:

$$C_0 = 0;$$

$$C_n = n + 1 + \frac{2}{n} \sum_{k=0}^{n-1} C_k$$

can try $C_1 = 2$, $C_2 = 5$, $C_3 = \frac{26}{3}$, yuck!

$$n C_n = n^2 + n + 2 \sum_{k=0}^{n-1} C_k \qquad n \leftarrow n-1$$

subtract

$$(n-1) C_{n-1} = (n-1)^2 + (n-1) + 2 \sum_{k=0}^{n-2} C_k$$

$$n C_n - (n-1) C_{n-1} = 2n + 2 C_{n-1}$$

$$C_0 = 0$$

$$n C_n = (n+1) C_{n-1} + 2n$$

We apply a summation factor

$$a_n = n, \quad b_n = n+1, \quad c_n = 2n$$

$$S_n = \frac{a_{n-1} a_{n-2} \cdots a_1}{b_n \cdots b_2} = \frac{(n-1)(n-2)\cdots 1}{(n+1)(n)(n-1)\cdots 3}$$

$$= \frac{2}{n(n+1)}$$

$$C_n = 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1}$$

This is related to the "harmonic number"

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

$$\sum_{k=1}^{n} \frac{1}{k+1} = \sum_{1 \le k \le n} \frac{1}{k+1} = \sum_{1 \le k-1 \le n} \frac{1}{k}$$

$$= \sum_{2 \le k \le n+1} \frac{1}{k} = H_n - \frac{1}{1} + \frac{1}{n+1}$$

$$= H_n - \frac{n}{n+1}$$

$$C_n = 2(n+1)\left(H_n - \frac{n}{n+1}\right) = 2(n+1)H_n - 2n$$

Manipulation of Sums

$$\sum_{k \in K} c\, a_k = c \sum_{k \in K} a_k \quad \text{(distributive)}$$

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k \quad \text{(associative)}$$

$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}$$ (commutative / reordering)

$p(\,)$ is a permutation

$$S = \sum_{0 \le k \le n} (a + bk)$$ replace k by $n-k$

$$= \sum_{0 \le n-k \le n} (a + b(n-k)) = \sum_{0 \le k \le n} a + bn - bk$$

$$2S = \sum_{0 \le k \le n} (a + bk + a + bn - bk)$$

$$= \sum_{0 \le k \le n} (2a + bn) = (2a + bn) \sum_{0 \le k \le n} 1$$

$$= (2a + bn)(n+1)$$

$$\text{So} \quad S = a(n+1) + b\,\frac{n(n+1)}{2}$$

As for permutations, consider

$$\sum_{\substack{k \in K \\ k \text{ even}}} a_k = \sum_{\substack{n \in K \\ n \text{ even}}} a_n = \sum_{\substack{2k \in K \\ 2k \text{ even}}} a_{2k} = \sum_{2k \in K} a_{2k}$$

$$\sum_{k\in K} a_k + \sum_{k\in K'} a_k = \sum_{k\in K\cup K'} a_k + \sum_{k\in K\cap K'} a_k$$

We use this to combine two almost distinct index sets, e.g.

$$\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{\substack{k=1 \\ k\neq m}}^{n} a_k, \quad 1\le m\le n.$$

Splitting off a term is the basis of perturbation method:

$$S_n = \sum_{0\le k\le n} a_k \quad \text{is unknown}$$

Write S_{n+1} two different ways

$$S_{n+1} = S_n + a_{n+1} = a_0 + \sum_{0\le k\le n} a_{k+1}$$

and try to get a formula for S_n out of it.

$$S_n = \sum_{0\le k\le n} a x^k$$

$$S_n + a x^{n+1} = a x^0 + \sum_{0 \le k \le n} a x^{k+1} = a x^0 + x S_n$$

$$S_n (1-x) = a - a x^{n+1} \Rightarrow S_n = a \frac{1 - x^{n+1}}{1 - x}, \; x \neq 1$$

Note: when $x = 1$ $\quad S_n = a \cdot (n+1)$

Another perturbation

$$S_n = \sum_{0 \le k \le n} k 2^k$$

$$S_{n+1} = S_n + (n+1) 2^{n+1} = \sum_{0 \le k \le n} (k+1) 2^{k+1}$$

$$\sum_{0 \le k \le n} k 2^{k+1} + \sum_{0 \le k \le n} 2^{k+1} =$$

$\uparrow$ $\qquad$ $\uparrow$ $\qquad$ missing "2^0"

$2 S_n \qquad \frac{1 - 2^{n+2}}{1 - 2} - 1 = 2^{n+2} - 2$

$$2 \cdot 2^{n+1}$$

$$S_n + (n+1) 2^{n+1} = 2 S_n + 2^{n+2} - 2$$

$$S_n = (n-1) 2^{n+1} + 2$$

Generally: $\sum_{k=0}^{n} k x^k = \frac{x - (n+1) x^{n+1} + n x^{n+2}}{(1-x)^2}$

as $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} = f(x)$

$f'(x) = \sum_{k=0}^{n} k x^{k-1} = \frac{1}{x} \sum_{k=0}^{n} k x^k$, so

$= x f'(x) = x \frac{1-(n+1)x^n + n x^{n+1}}{(1-x)^2}$.

Multiple Sums

$$\sum_{1 \le i,j \le 3} a_i b_j = \begin{matrix} a_1 b_1 + a_1 b_2 + a_1 b_3 + \\ a_2 b_1 + a_2 b_2 + a_2 b_3 + \\ a_3 b_1 + a_3 b_2 + a_3 b_3 \end{matrix} \left[\begin{matrix} \text{"outer} \\ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{product"} \\ (a_1, a_2, a_3) \end{matrix} \right]$$

Also $\sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k} [P(j,k)]$, also

$$\sum_j \sum_k a_{j,k} [P(j,k)] = \sum_j \left(\sum_k a_{j,k} [P(j,k)] \right)$$

for finite sums, no problem with interchange.

$$\sum_{1 \le j,k \le 3} a_j b_k = \left(\sum_{j=1}^{3} a_j \right) \left(\sum_{k=1}^{3} b_k \right)$$

gen. distr. law

Simple interchange

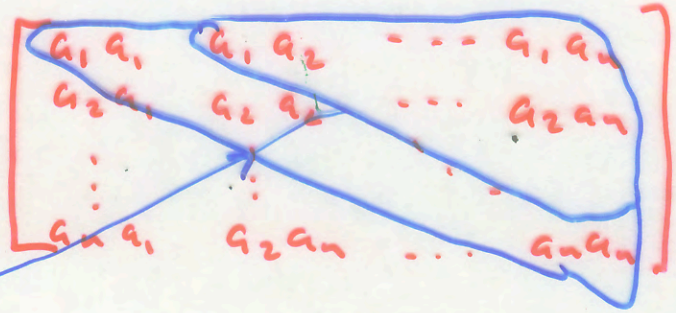
$$\sum_{j\in J}\sum_{k\in K} a_{j,k} = \sum_{\substack{j\in J\\ k\in K}} a_{j,k} = \sum_{k\in K}\sum_{j\in J} a_{j,k}$$

what if $K = K(j)$?

$$\sum_{j=1}^{n}\sum_{k=j}^{n} a_{j,k} = \sum_{1\le j\le k\le n} a_{j,k} = \sum_{k=1}^{n}\sum_{j=1}^{k} a_{j,k}$$

$$[1\le j\le n][j\le k\le n] = [1\le j\le k\le n] = [1\le k\le n][1\le j\le k]$$

Consider



$$S_{\triangledown} = \sum_{1\le j\le k\le n} a_j a_k = \sum_{1\le k\le j\le n} a_k a_j$$

$$= \sum_{1\le k\le j\le n} a_j a_k = S_{\triangle}$$

$$2S_{\triangledown} = S_{\triangledown} + S_{\triangle} = \sum_{1\le j,k\le n} a_j a_k + \sum_{1\le j=k\le n} a_j a_k$$

$$\sum_{1\le j,k\le n} a_j a_k = \left(\sum_{k=1}^{n} a_k\right)^2, \quad \sum_{1\le j=k\le n} a_j a_k = \sum_{k=1}^{n} a_k^2, \text{ so}$$

$$S_\triangle = S_\square = \frac{1}{2}\left[\left(\sum_{k=1}^{n} a_k\right)^2 + \sum_{k=1}^{n} a_k^2\right]$$

Consider $S = \sum_{1\le j<k\le n} (a_k - a_j)(b_k - b_j)$

↑ to avoid zeros

by using symmetry again we get

$$2S = \sum_{1\le j,k\le n} (a_j - a_k)(b_j - b_k) - \sum_{1\le j=k\le n} (a_j - a_k)(b_j - b_k)$$

$$\text{(the second sum)} = 0$$

$$= \sum_{1\le j,k\le n} (a_j b_j) - (a_j b_k) - (a_k b_j) + (a_k b_k)$$

$$= 2\sum_{1\le j,k\le n} a_k b_k - 2\sum_{1\le j,k\le n} a_j b_k$$

$$= 2n \sum_{1 \le k \le n} a_k b_k - 2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \text{ which gives}$$

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) = n \sum_{k=1}^{n} a_k b_k - \sum_{1 \le j < k \le n} (a_k - a_j)(b_k - b_j) \quad (*)$$

note if $a_1 \le a_2 \le \cdots \le a_n$ & $b_1 \le b_2 \le \cdots \le b_n$

$(*)$ is negative so throwing it away $\le$

if the b_i's are monotonically decreasing $\ge$

Consider $S_n = \sum_{1 \le j < k \le n} \frac{1}{k-j}$

$S_1 = 0$; $S_2 = 1$, $S_3 = \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{3-2} = \frac{5}{2}$

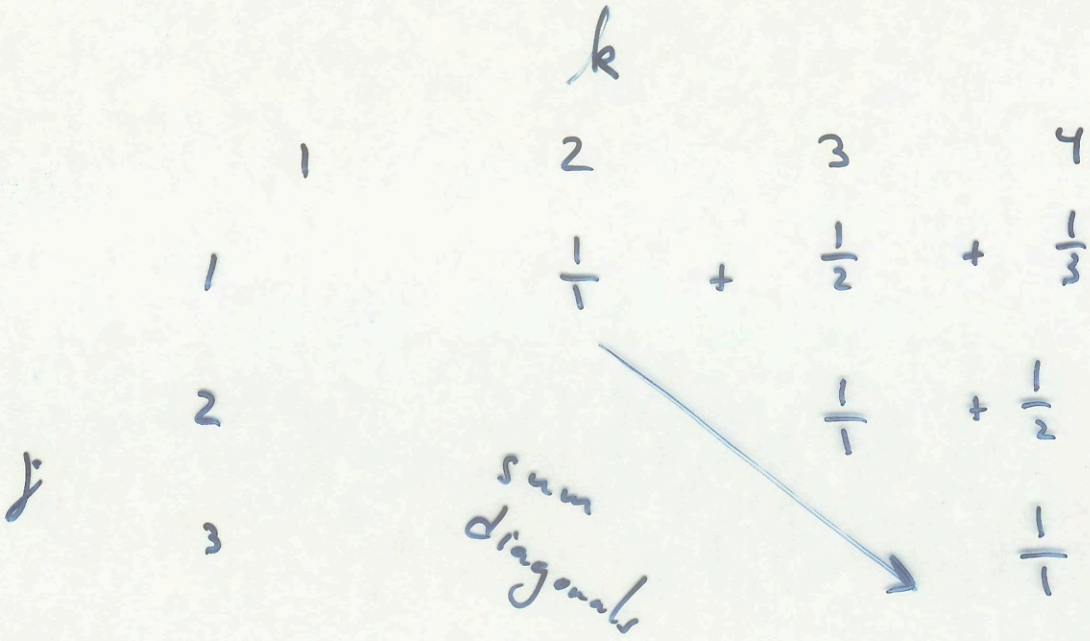
if we sum first on k or j we get

$S_n = \sum_{0 \le k < n} H_k$ which we don't know

$$S_n = \sum_{1 \le j < k \le n} \frac{1}{k-j}$$ replace k by $k+j$

$$= \sum_{1 \le j < k+j \le n} \frac{1}{k} = \sum_{1 \le k \le n} \sum_{1 \le j \le n-k} \frac{1}{k}$$

$$= \sum_{1 \le k \le n} \frac{n-k}{k} = n \sum_{1 \le k \le n} \frac{1}{k} - n = n H_n - n$$



General Methods

Consider $\square_n = \sum_{0 \le k \le n} k^2$, $n \ge 0$

Method 0: Look it up

Abramowitz & Stegun

Hansen's table

Sloane's handbook

Method 1: Guess the answer, prove by induction

$$\square_n = \frac{n\left(n+\frac{1}{2}\right)(n+1)}{3}, \quad \square_0 = 0 = \frac{0\cdot\left(0+\frac{1}{2}\right)(0+1)}{3} \checkmark$$

$$\square_n = \square_{n-1} + n^2 = \frac{(n-1)\left(n-\frac{1}{2}\right)(n)}{3} + n^2$$

$$= \frac{n^3 - \frac{3}{2}n^2 + \frac{n}{2}}{3} + n^2$$

$$= \frac{n^3 - \frac{3}{2}n^2 + \frac{n}{2} + 3n^2}{3} = \frac{n^3 + \frac{3}{2}n^2 + \frac{n}{2}}{3}$$

$$= \frac{n\left(n^2 + \frac{3}{2}n + \frac{1}{2}\right)}{3} = \frac{n\left(n+\frac{1}{2}\right)(n+1)}{3} \quad \square$$

Method 2: Perturb the sum

$$\square_{n+1} = \square_n + (n+1)^2 = \sum_{0 \le k \le n} (k+1)^2$$

$$= \sum_{0 \le k \le n} (k^2 + 2k + 1) = \square_n + 2\sum_{0 \le k \le n} k + (n+1)$$

Hey "$\square_n$" cancel out, but we get

$$2\sum_{k=1}^{n} k = (n+1)^2 - (n+1)$$

Consider $\square_n = \sum_{0 \le k \le n} k^3$

$$\square_n + (n+1)^3 = \square_{n+1} = \sum_{0 \le k \le n} (k+1)^3$$

$$= \sum_{0 \le k \le n} (k^3 + 3k^2 + 3k + 1) = \square_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1)$$

So $$3\square_n = (n+1)^3 - \frac{3(n+1)n}{2} - (n+1)$$

$$= (n+1)\left(n^2 + 2n + 1 - \frac{3n}{2} - 1\right) = (n+1)\left(n^2 + \frac{n}{2}\right)$$

Method 3: Build a repertoire

$R_0 = \alpha$;

$R_n = R_{n-1} + \beta + \gamma n + \delta n^2$, $n > 0$

$R_n = A_{(n)}\alpha + B_{(n)}\beta + C_{(n)}\gamma + D_{(n)}\delta$

already have $A_{(n)}$, $B_{(n)}$, and $C_{(n)}$. what

if $R_n = n^3 \Rightarrow \alpha = 0 \quad \beta = 1 \quad \gamma = -3 \quad \delta = 3$

$$3D_{(n)} - 3C_{(n)} + B_{(n)} = n^3$$

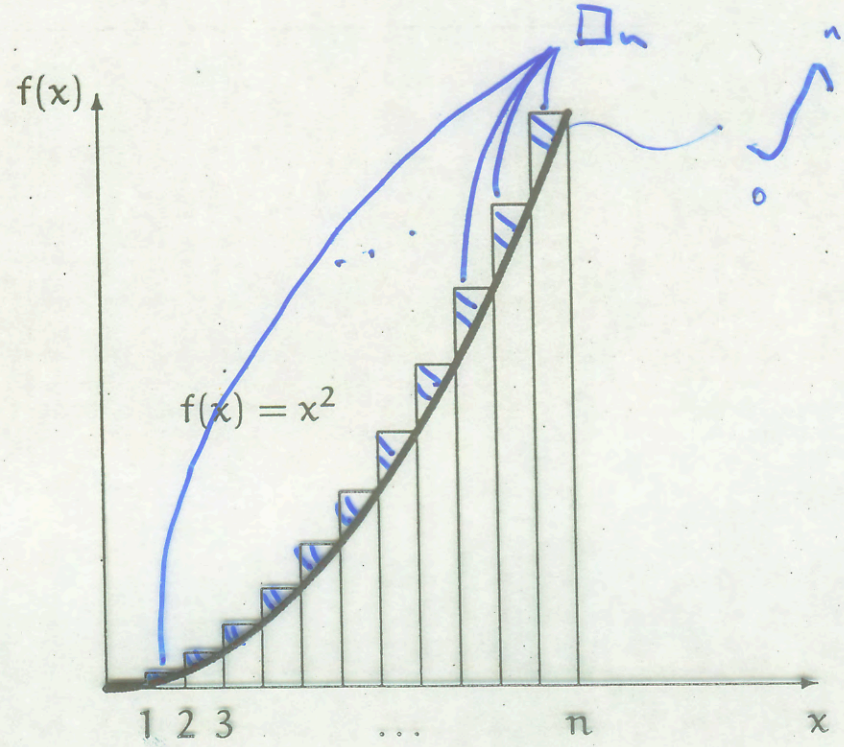
$\square_n = R_n$ when $\alpha = \beta = \gamma = 0$; $\delta = 1$

So $\square_n = D(n)$

$$3D(n) = n^3 + 3C(n) - B(n)$$

$$= n^3 + 3\,\frac{(n+1)n}{2} - n = n\left(n+\tfrac{1}{2}\right)(n+1)$$

Method 4: Replace sums by integrals



$$\int_0^n x^2\,dx = \frac{n^3}{3}$$

$$E_n = \square_n - \int_0^n x^2\,dx = \square_n - \frac{n^3}{3}$$

$$= \square_{n-1} + n^2 - \frac{n^3}{3} = E_{n-1} + \frac{(n-1)^3}{3} - \frac{n^3}{3} + n^2$$

$$E_{n-1} + \frac{(n-1)^3}{3} = E_{n-1} + \frac{n^3 - 3n^2 + 3n - 1 - n^3 + 3n^2}{3}$$

$$= E_{n-1} + n - \frac{1}{3}$$

Another approach:

$$\square_n - \int_0^n x^2\,dx = \sum_{k=1}^{n}\left(k^2 - \int_{k-1}^{k} x^2\,dx\right)$$

$$= \sum_{k=1}^{n}\left(k^2 - \frac{k^3 - (k-1)^3}{3}\right)$$

$$= \sum_{k=1}^{n}\left(k^2 - \frac{k^3 - (k^3 - 3k^2 + 3k - 1)}{3}\right)$$

$$= \sum_{k=1}^{n}\left(k - \frac{1}{3}\right) = \frac{n(n+1)}{2} - \frac{1}{3}n \qquad \text{so}$$

$$\square_n = \frac{n^3}{3} + \frac{n(n+1)}{2} - \frac{1}{3}n$$

$$= \frac{2n^3 + 3n^2 + 3n - 2n}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

$$= \frac{2n\left(n^2 + \frac{3}{2}n + \frac{1}{2}\right)}{6} = \frac{n\left(n+\frac{1}{2}\right)(n+1)}{3}$$ ✓

Method 6: Use finite calculus (next)

Method 7: Use generating functions (chapters later)

Finite and Infinite Calculus

Derivative (infinite calculus) $Df(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Define the difference operator, Δ, as

$$\Delta f(x) = f(x+1) - f(x) \qquad (h=1)$$

$D(x^m) = m x^{m-1}$, for example

$\Delta(x^3) = (x+1)^3 - x^3 = 3x^2 + 3x + 1$

so the normal "mth power" is messy for Δ, but not this "weird" "mth power"

$$x^{\underline{m}} = \overbrace{x(x-1)(x-2)\cdots(x-m+1)}^{m \text{ factors}}$$

"x to the m falling" $= \prod_{k=1}^{m} (x-k+1)$, also we have

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

"x to the m rising" $= \prod_{k=1}^{m} (x+k-1)$

or falling/rising factorial powers

note $n! = n^{\underline{n}} = 1^{\overline{n}}$

$$\Delta(x^{\underline{m}}) = (x+1)^{\underline{m}} - x^{\underline{m}}$$

(common)

$$= (x+1)(x)(x-1)\cdots(x-m+2) - x\cdots(x-m+2)(x-m+1)$$

$$= [(x+1)-(x-m+1)]\; x(x-1)\cdots(x-m+2)$$

$$= m\, x^{\underline{m-1}}$$

Recall the Fundamental Theorem of Calculus:

$$g(x) = D f(x) \iff \int g(x)\,dx = f(x) + C$$

Here, with finite calculus we have an inverse to Δ, it's summation:

$$g(x) = \Delta f(x) \iff \sum g(x)\,\delta x = f(x) + C$$

(indefinite sum of $g(x)$)

C here is any function constant on the integers, i.e. $p(x) = p(x+1) = c$

Recall the definite integral $\int_a^b g(x)\,dx =$

$$f(x)\Big|_a^b = f(b) - f(a)$$

if $g(x) = \Delta f(x)$, then

$$\underbrace{\sum_a^b g(x)\,\delta x}_{\text{what really is this}} = f(x)\Big|_a^b = f(b) - f(a)$$

Assume $g(x) = \Delta f(x) = f(x+1) - f(x)$

$$\sum_a^a g(x)\,\delta x = f(a) - f(a) = 0 \quad \text{should hold}$$

$$\sum_a^{a+1} g(x)\,\delta x = f(a+1) - f(a) = g(a)$$

$$\sum_a^{b+1} g(x)\,\delta x - \sum_a^b g(x)\,\delta x = f(b+1) - f(a) - [f(b) - f(a)]$$

$$= f(b+1) - f(b) = g(b).$$

thus $\displaystyle\sum_a^b g(x)\,\delta x = \sum_{h=a}^{b-1} g(h) = \sum_{a \le h < b} g(h)$

$g(h)$

$$\sum_{a \le h < b} (f(h+1) - f(h)) = f(b) - f(a)$$

↖ telescoping sum.

$$\sum_a^b g(x)\,\delta x = f(b) - f(a) = -(f(a) - f(b))$$

$$= -\sum_b^a g(x)\,\delta x$$

$$\sum_a^b g(x)\,\delta x + \sum_b^c g(x)\,\delta x = \sum_a^c g(x)\,\delta x$$

Example: $$\sum_{0 \le k < n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Bigg|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

let $m = 1$

$$\sum_{0 \le k < n} k^{\underline{1}} = \sum_{0 \le k < n} k = \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$$

$$k^2 = k^{\underline{2}} + k^{\underline{1}} = k(k-1) + k$$

$$\square_{n-1} = \sum_{0 \le k < n} k^2 = \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} = \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2}$$

$$= \frac{1}{3}\, n\left(n - \tfrac{1}{2}\right)(n-1)$$

$$k^3 = k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}, \quad + \text{h.o.t.}$$

$$\text{cube}_{n-1} = \sum_{0 \le k < n} k^3 = \frac{k^{\underline{4}}}{4} + k^{\underline{3}} + \frac{k^{\underline{2}}}{2}\Bigg|_0^n$$

In general we have the "factorial binomial thm."

$(x+y)^{\underline{m}}$ can be written via binomial thm.

what about $x^{\underline{m}}$, $m < 0$?

$x^{\underline{3}} = x(x-1)(x-2)$

$x^{\underline{2}} = x(x-1)$ ↙ divide by $(x-2)$

$x^{\underline{1}} = x$ ↙ divide by $x-1$

$x^{\underline{0}} = 1$ ↙ divide by x

$x^{\underline{-1}} = ?$ "divide by $x+1$"

$= \frac{1}{x+1}$

$x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$, $x^{\underline{-3}} = \frac{1}{(x+1)(x+2)(x+3)}$

$x^{\underline{-m}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$, for $m > 0$

Thus $x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$, e.g.

$x^{\underline{2+3}} = x^{\underline{2}}(x-2)^{\underline{3}}$

$x^{\underline{2-3}} = x^{\underline{2}}(x-2)^{\underline{-3}} = x\cdot(x-1)\cdot\frac{1}{(x-1)x(x+1)} = \frac{1}{x+1}$

Thus $\sum_a^b x^{\underline{m}}\,\delta x = \frac{x^{\underline{m+1}}}{m+1}\Big|_a^b$, $m \neq -1$ $= x^{\underline{-1}}$

What about $m=-1$? $x^{\underline{-1}} = \frac{1}{x+1} = \Delta f(x)$?

$f(x) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{x}$ $\quad f(x+1) = \frac{1}{x+1} + f(x)$

$f(x+1) - f(x) = \frac{1}{x+1}$

$f(n) = H_n$, n an integer, note that

$H_x - \ln x \approx 0.577 + \frac{1}{2x}$, so it is analogous:

$$\sum_a^b x^{\underline{m}}\,\delta x = \begin{cases} \left.\frac{x^{\underline{m+1}}}{m+1}\right|_a^b & m \neq -1 \\ \left. H_x \right|_a^b & m = -1 \end{cases}$$

What is the analog of the exponential? What $f(x)$ has $\Delta f(x) = f(x)$ $\quad f(x+1) = 2f(x)$

$\Delta c^x = c^{x+1} - c^x = (c-1)c^x$, so we have

$$\sum_{a \le k < b} c^x = \sum_a^b c^x \delta x = \left.\frac{c^x}{c-1}\right|_a^b = \frac{c^b - c^a}{c-1}, \quad c \neq 1$$

Other analogies: no chain rule, change of variables is hard, but there are others.

Here is a table summarizing the results.

Table 55 What's the difference?

$f = \Sigma g$	$\Delta f = g$	$f = \Sigma g$	$\Delta f = g$
$x^{\underline{0}} = 1$	0	2^x	2^x
$x^{\underline{1}} = x$	1	c^x	$(c-1)c^x$
$x^{\underline{2}} = x(x-1)$	$2x$	$c^x/(c-1)$	c^x
$x^{\underline{m}}$	$mx^{\underline{m-1}}$	cf	$c\Delta f$
$x^{\underline{m+1}}/(m+1)$	$x^{\underline{m}}$	$f+g$	$\Delta f + \Delta g$
H_x	$x^{\underline{-1}} = 1/(x+1)$	fg	$f\Delta g + Eg\Delta f$

Summation by parts (integration by parts)

$D(uv) = uDv + vDu$ (product rule)

$\int u\,Dv = uv - \int v\,Du$ (int. by parts)

$\Delta(u(x)v(x)) = u(x+1)v(x+1) - u(x)v(x)$

$(+ u(x)v(x+1) - u(x)v(x+1))$

$= u(x)\Delta v(x) + v(x+1)\Delta u(x)$

Define the shift operator $Ef(x) = f(x+1)$

$\Delta(uv) = u\Delta v + Ev\Delta u$

$\sum u\Delta v = uv - \sum Ev\Delta u$

$\sum_{k=0}^{n} k 2^k$: $u(x) = x$, $\Delta v(x) = 2^x$,

$\Delta u(x) = 1$, $v(x) = 2^x$, $Ev(x) = 2^{x+1}$

$$\sum x 2^x \delta x = x 2^x - \sum 2^{x+1} \delta x$$

$$= x 2^x - 2^{x+1} + C$$

$$\sum_{k=0}^{n} k 2^k = \left(x 2^x - 2^{x+1} \right) \Big|_0^{n+1}$$

$$= \left[(n+1) 2^{n+1} - 2^{n+2} \right] - \left[0 \cdot 2^0 - 2^1 \right]$$

$$= 2^{n+1}(n-1) + 2$$

Consider $\sum_{0 \le k < n} k H_k$, summation by parts

$u(x) = H_x$, $\Delta v(x) = x = x^{\underline{1}}$,

$\Delta u(x) = x^{\underline{-1}}$, $v(x) = \frac{x^{\underline{2}}}{2}$, $Ev(x) = \frac{(x+1)^{\underline{2}}}{2}$

$$\sum x H_x \delta x = \frac{x^{\underline{2}}}{2} H_x - \sum \frac{(x+1)^{\underline{2}}}{2} x^{\underline{-1}} \delta x$$

$$= \frac{x^{\underline{2}}}{2} H_x - \frac{1}{2} \sum (x+1) x \frac{1}{(x+1)} \delta x$$

$$= \frac{x^{\underline{2}}}{2} H_x - \frac{x^{\underline{2}}}{4} + C$$

$$\sum_{0 \le k < n} k H_k = \sum_0^n x H_x \delta_x = \frac{n^2}{2}\left(H_n - \frac{1}{2}\right)$$

Infinite Sums

Some of the techniques we have used continue to work, some don't.

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

$$2S = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots = 2 + S$$

$$\Rightarrow S = 2 \quad \text{seems to work}$$

$$T = 1 + 2 + 4 + 8 + 16 + \cdots$$

$$2T = 2 + 4 + 8 + 16 + 32 + \cdots = T - 1$$

$$\Rightarrow T = -1 \; ? \quad \text{perhaps } T = \infty$$

Consider $a_k \ge 0$, K an index set, then if

$$\sum_{k \in F} a_k \le A \quad \forall \text{ finite } F \subset K$$

this should provide an answer by making A as small as possible

Usually $K = \{0, 1, 2, \dots\} = \mathbb{N}$ "natural nums."

$$\sum_{k \geq 0} a_k = \lim_{n \to \infty} S_n, \quad S_n = \sum_{k=0}^{n} a_k,$$

if we can bound this, and $a_k \geq 0$, there must be a limit.

$$\sum_{k \geq 0} x^k = \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} (1-x)^{-1}, & 0 \leq x < 1 \\ \infty, & x \geq 1 \end{cases}$$

S, T are of this form. Now consider

$$\sum_{k \geq 0} \frac{1}{(k+1)(k+2)} = \sum_{k \geq 0} k^{-2} = \lim_{n \to \infty} \sum_{0}^{n} x^{-2} \, \delta x$$

$$= \lim_{n \to \infty} \frac{x^{-1}}{-1} \Bigg|_0^n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

What about when some a_k's are negative?

$$\sum_{k \geq 0} (-1)^k = 1 - 1 + (-1 + \cdots$$

$$(\ ) \ (\ ) \qquad = 0 \ ?$$

$$(\ ) \ (\ ) \qquad = 1 \ ?$$

Also if $x = -1$ in $\sum_{k \geq 0} x^k = \frac{1}{1-x} = \frac{1}{2}$?

$$\sum_{k \in K} a_k = \sum_{k \in K} a_k^+ - \sum_{k \in K} a_k^-$$

a_k^+: positive; a_k^-: negative

then investigate sep.

If $A^\pm = \sum_{k \in K} a_k^\pm$ and $A^\pm < \infty$ then

$\sum_{k \in K} a_k$ converges absolutely to $A^+ - A^-$

The problem arises when A^+ and/or A^- is infinite. Then reordering can give you any value you wish, pretty much.

Expand and Contract (Method 5)

$$\square_n = \sum_{1 \le k \le n} k^2$$

$$= \sum_{1 \le j \le k \le n} k \qquad (k \text{ copies of } k)$$

$$= \sum_{1 \le j \le n} \sum_{j \le k \le n} k \qquad (*)$$

$$[1 \le j \le k \le n] = [1 \le j \le n][j \le k \le n] \qquad \text{more useful}$$
$$= [1 \le k \le n][1 \le j \le k]$$

$$\sum_{j \le k \le n} k = \left(\frac{\text{first} + \text{last}}{2}\right) \#\text{ of terms}$$

$$= \left(\frac{j+n}{2}\right)(n-j+1)$$

$$= \frac{1}{2}\left[n(n+1) + j - j^2\right]$$

$$(*) = \frac{1}{2} \sum_{1 \le j \le n} \left[n(n+1) + j - j^2\right] =$$

$$= \frac{1}{2} n^2(n+1) + \frac{1}{4} n(n+1) - \frac{1}{2} \square_n$$

$$= \frac{1}{2} n(n+1)\left[n+\frac{1}{2}\right] - \frac{1}{2} \square_n = \square_n$$

$$\frac{3}{2} \square_n = \frac{n(n+\frac{1}{2})(n+1)}{2}$$

$$\square_n = \frac{n(n+\frac{1}{2})(n+1)}{3}$$