Concrete Mathematics

Continuous

Discrete

The mathematical basis for advanced Computer Science (minus the graph theory)

See syllabus

Recurrences

What should you already know?

Solving \( T(n) = aT\left(\frac{n}{b}\right) + f(n) \)
Methods

a) Repeated substitution + inspection
b) Tree method (graphical)
c) Master Theorem

\[ T(cn) = a \cdot T \left( \frac{n}{b} \right) + f(cn) \]

# of subproblems of size \( \frac{n}{b} \)

recombination cost

Compare \( n \log_b a \) to \( f(cn) \)

the bigger one wins, in case of a tie, multiply by \( \log n \)
Master Theorem (Method)

Case 1: if \( f(n) = \Theta(n \log_a n^c) \), \( c > 0 \)
then \( T(n) = \Theta(n \log_a n) \)

Case 2: if \( f(n) = \Theta(n \log_a n^c) \), then

\[
T(n) = \Theta(f(n) \log n) = \Theta(n \log_a n \log n)
\]

Case 3: if \( f(n) = \Omega(n \log_a n^{c+\epsilon}) \), \( c > 0 \)
and \( a f(n/b) \leq c \cdot n \), \( 0 < c < 1 \),
then \( T(n) = \Theta(f(n)) \)
Some Problems

Tower of Hanoi

\[ T_n = \text{minimum number of moves required to move a tower of } n \text{ disks} \]

\[ (T(n)?) \]

By playing around

\[ T_1 = 1 \quad (\text{duh!}) \]

\[ T_2 = 3 \quad \text{by inspection} \]

With a tower of \( n \), we must move \( n-1 \) from A to blank, biggest from A to B
and then the \( n-1 \) from blank to B.

This means \( T_n \leq 2T_{n-1} + 1 \quad n > 0 \)

Can also argue

\[ T_n \geq 2T_{n-1} + 1 \quad n > 0 \]

We must eventually move the largest disk, to do this the \( n-1 \) smaller must be on a single spindle, taking \( T_{n-1} \) in the best case. We then must move the disk to B and continue.

\( T_0 = 0 \)

Note: Recurrences have two parts:

\( T_0 = 0 \) boundary condition

\( T_n = 2T_{n-1} + 1 \quad n > 0 \) (iteration)

Note: This does not fit into the master's method rubrik!
\[ T(n) = 2T(n-1) + 1 \]
\[ = 2 \cdot T(n \cdot \frac{n-1}{n}) + 1 \]

\[ a \quad \text{？} \quad b \quad \text{no, b must be constant} \]

How do we solve this?

\[ T_0 = 0 \]
\[ T_1 = 1 \]
\[ T_2 = 2 \cdot 1 + 1 = 3 \]
\[ T_3 = 2 \cdot 3 + 1 = 7 \]
\[ T_4 = 2 \cdot 7 + 1 = 15 \]
\[ T_5 = 2 \cdot 15 + 1 = 31 \]

Computer Scientists should recognize the pattern:

\[ T_n = 2^n - 1 \quad n \geq 0 \]

This is a guess, must prove by induction.

\[ T_0 = 0 \quad 0 = n_0, \text{ the basis} \]
Assume $T_{n+1} = 2^{n+1} - 1$

$$T_n = 2 \cdot T_{n-1} + 1 = 2 \cdot (2^{n-1} - 1) + 1$$
$$= 2^n - 2 + 1 = 2^n - 1$$ \hfill \square$$

Note: $U_0 = 1$

$$U_n = 2 \cdot U_{n-1}, \quad n > 0$$

$$U_n = T_n + 1 \quad \Rightarrow \quad T_n = U_n - 1$$

$$T_0 + 1 = 1$$

$$T_n + 1 = 2 \cdot T_{n-1} + 1 + 1$$
$$= 2 \cdot T_{n-1} + 2 = 2 \cdot (T_{n-1} + 1)$$

$$U_n = 2^n \quad \text{so} \quad T_n = 2^n - 1$$

Second warmup problem

Lines in the plane

Problem posed by Steiner in 1826.
Compute $L_n$, the maximum number of regions the plane (pizza) can be cut by $n$ straight lines:

$L_0 = 1 \quad L_1 = 2 \quad L_2 = 4 \quad L_3 = ?$

\[
\begin{array}{c|c|c|c}
1 & 2 & 3 \\
\hline
4 & 5 & 6
\end{array}
\]

\[
\begin{array}{c|c|c}
1b & 1a & 2 \\
\hline
4b & 4a & 3a \\
\hline
& 3b &
\end{array}
\]

$L_3 = 7$ note: above fails because the new line intersects the old lines at only a single point.
What if our construction always adds a line that intersects each of the other lines at least once. Doable? Yes by ensuring the new line is not parallel to the existing lines.

\[ L_n = L_{n-1} + n \quad n \geq 0 \]

\[ L_0 = 1 \quad (B.C.) \]

\[ L(n) = L\left(n \cdot \frac{n-1}{n}\right) + n \]

\[ L_0 = 1 \]
\[ L_1 = 1 + 1 = 2 \]
\[ L_2 = 2 + 2 = 4 \]
\[ L_3 = 4 + 3 = 7 \quad \text{this is not so helpful} \]

\[ L_n = L_{n-1} + n \]
\[ = L_{n-2} + n + (n-1) \]
\[ = L_{n-3} + n + (n-1) + (n-2) \]
\[ = L_0 + n + (n-1) + (n-2) + \ldots + 1 \]
\[ = 1 + 5n \]

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\[ S_n = \sum_{i=1}^{n} x_i = \frac{n(n+1)}{2} \]

\[ L_n = \frac{n(n+1)}{2} + 1 \]

**Proof:** \( L_0 = 1 \) holds

\[ L_{n-1} = \frac{(n-1)(n-1+1)}{2} + 1 = \frac{n(n-1)}{2} + 1 \]

\[ L_n = \frac{n(n-1)}{2} + 1 + n = \frac{n^2 - n}{2} + \frac{2n}{2} + 1 \]

\[ = \frac{n^2 + n}{2} + 1 = \frac{n(n+1)}{2} + 1 \]

\[ \square \]

Consider

\[ \exists \begin{array}{c} 2 \end{array} \]

\[ \exists \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \]

\[ \exists \begin{array}{c} 0 \end{array} \begin{array}{c} 7 \end{array} \begin{array}{c} 4 \end{array} \]

\[ \exists \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \]

\[ \exists \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \]

\[ \exists \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \begin{array}{c} 7 \end{array} \]

\[ \exists \begin{array}{c} -10 \end{array} \]
This is like the previous problem but with two lines and merging.

\[ Z_n = L Z_n - 2n \]

\[ \text{double lines} \quad \text{loss from merging} \]

\[ Z_n = \frac{2n (2n+1)}{2} + 1 - 2n \]

\[ = 2n^2 + n + 1 - 2n \]

\[ = 2n^2 - n + 1 \quad , \quad n \geq 0 \]

\[ Z_n = \Theta(n^2) \]

\[ L_n = \Theta(n^2) \]

not really surprising.
The Josephus Problem

eliminate in order 2, 4, 6, 8, 10, 3, 7, 1, 9
with 5 surviving.

\[ J(10) = 5 \text{ is implied } , \quad J(n) = \frac{n}{2} ? \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J(2n) )</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Since by convention we always start with 2, we always eliminate all the evens. Thus if we have \( 2n \) (an even number) of people

\[ J(2n) = 2 J(n) - 1 \]

just solve one problem twice a small and re-number.
So \( J(10) = 2J(5) - 1 = 2 \cdot 3 - 1 = 5 \)
checks out. Also
\[
J(20) = 2J(10) - 1 = 2 \cdot 5 - 1 = 9 \\
= 2(2J(5) - 1) - 1 \\
= 4J(5) - 3 = 4 \cdot 3 - 3 = 9
\]

Can prove (by induction):
\[
J(5, 2^m) = 2^{m+1} + 1
\]

basis: \( m = 0 \) \( J(5) = 3 = 2^{0+1} + 1 \) \( \checkmark \)

\[
J(5, 2^m) = 2 \cdot J(5, 2^{m-1}) - 1 \\
= 2 \cdot (2^m + 1) - 1 \\
= 2^{m+1} + 2 - 1 = 2^{m+1} + 1
\]

With \( 2n+1 \) people (odd)
we wipe out first all the even and then
\[\begin{array}{c}
\scriptsize 2n+1 \\
\scriptsize 2n-1 \\
\scriptsize \ldots \\
\scriptsize 7 \\
\scriptsize 5 \\
\scriptsize 3
\end{array}\]

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Thus

\[ J(2n+1) = 2J(n) + 1 \]

( Check it yourself)

Combining we get the recurrence:

\[
\begin{align*}
J(1) &= 1 \quad (B.C.) \\
J(2n) &= 2J(n) - 1 \\
J(2n+1) &= 2J(n) + 1
\end{align*}
\]

\( n \geq 1 \)

Can use this as a "fast leap-ahead"

\[
\begin{array}{cccccccccc}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
J(n) & 1 & 1 & 3 & 1 & 3 & 5 & 7 & 13 & 8 & 13 & 14 & 15 & 17 & 18 & 19 & 1 \\
\hline
\end{array}
\]

Note the grouping by powers-of-two!

\[ n = 2^m + \ell, \quad 0 \leq \ell < 2^m \]

\( \uparrow \) largest power-of-two in \( n \)

Seems \( J(n) = 2\ell + 1 \)
Induction on $m$:

$m = 0 \quad n = 2^0 + l = 1 \quad \therefore l = 0$

$J(1) = 2 \cdot 0 + 1 = 1 \checkmark$

Since the recurrence has two formulae, we must use both:

$J(2^m + l) = 2 \cdot J(2^{m-1} + \frac{l}{2}) - 1$

$= 2 \cdot \left(2^{\frac{l}{2}} + 1\right) - 1 \quad \text{even case}$

$= 2 \cdot l + 1$

$= 2 \cdot J(2^{m-1} + \frac{l-1}{2}) + 1$

$= 2 \cdot \left(2^{\frac{l-1}{2}} + 1\right) + 1 \quad \text{odd case}$

$= 2 \cdot (l - 1) + 2 + 1$

$= 2 \cdot l + 1$

This is a closed-form solution.

Consider

$n = (b_m b_{m-1} \ldots b_1 b_0)_{2}$ in base 2
Note \( b_m = 1 \) (Why?)

\[
\begin{align*}
\ell &= (0 \ b_{m-1} \ b_{m-2} \ \ldots \ \ b_1 \ b_0)_2 \\
2\ell &= (b_{m-1} \ b_{m-2} \ \ldots \ \ b_1 \ b_0 \ 0)_2 \\
2\ell + 1 &= (b_{m-1} \ b_{m-2} \ \ldots \ \ b_1 \ b_0 \ 1)_2 \\
&\uparrow \ b_m
\end{align*}
\]

So, \( J((b_m \ \ldots \ b_0)_2) = (b_{m-1} \ \ldots \ b_0 \ b_m)_2 \)

is a one-bit left cyclic shift.

\[
\begin{align*}
h &= 100 = (1100100)_2 \\
J(n) &= (1001001)_2 = 64 + 9 + 1 = 73 \\
100 &= 64 + 36 \\
\ell &= 72 + 1 = 73
\end{align*}
\]

One oddity, \( n = 13 = (1101)_2 \)

\[
\begin{align*}
J((1101)_2) &= (1011)_2 \\
J((1011)_2) &= (0111)_2 \\
&\uparrow \text{ gets dropped}
\end{align*}
\]

This is because \( J(n) < n \).

If we iterate \( J(\cdot) \) what happens?
"We squeeze the 0's out" and eventually get a fixed point $J(n) = n$

with $n = (1111 \ldots 1)_2 = 2^{v(n)} - 1$

$v(n) = \text{pop\_count}(n)$ ($\#$/ of 1's in binary $n$).

What about the original conjecture:

$J(n) = \frac{n}{2}$ holds when?

$2l + 1 = \frac{1}{2} (2^m + l) = 2^{m-1} + l/2$

$\frac{3}{2} l = 2^{m-1} - 1$

$l = \frac{1}{3} (2^{m-2})$

If $l$ is an integer then $n = 2^m + l$ will solve this. $2^m - 2 \equiv 0 \pmod{3} \implies m$ odd

\[
\begin{array}{ccc}
110 & = & 6 \\
1010 & = & 10 \\
10010 & = & 18 \\
100010 & = & 34 \\
\end{array}
\]

If $m$ is odd then $n = 10_2 \rightarrow 2^m + l$ gives

<table>
<thead>
<tr>
<th>$m$</th>
<th>$l$</th>
<th>$n = 2^m + l$</th>
<th>$J(n) = \frac{n}{2}$</th>
<th>$n$ (base 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>10</td>
<td>5</td>
<td>1010</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>42</td>
<td>21</td>
<td>101010</td>
</tr>
<tr>
<td>7</td>
<td>42</td>
<td>120</td>
<td>-12</td>
<td>10101010</td>
</tr>
</tbody>
</table>
Generalize

\[ f(1) = \alpha \quad (1) \]
\[ f(2n) = 2f(n) + \beta \quad (-1) \]
\[ f(2n+1) = 2f(n) + \gamma \quad (1) \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>2</td>
<td>( 2\alpha + \beta )</td>
</tr>
<tr>
<td>3</td>
<td>( 2\alpha + \gamma )</td>
</tr>
<tr>
<td>4</td>
<td>( 4\alpha + 3\beta )</td>
</tr>
<tr>
<td>5</td>
<td>( 4\alpha + 2\beta + \gamma )</td>
</tr>
<tr>
<td>6</td>
<td>( 4\alpha + \beta + 2\gamma )</td>
</tr>
<tr>
<td>7</td>
<td>( 4\alpha + \gamma + 3\beta )</td>
</tr>
<tr>
<td>8</td>
<td>( 8\alpha + 7\beta )</td>
</tr>
<tr>
<td>9</td>
<td>( 8\alpha + 6\beta + \gamma )</td>
</tr>
</tbody>
</table>

\[ f(cn) = A(cn) \alpha + B(cn) \beta + C(cn) \gamma \]

where here:

\[ A(cn) = 2^m \]
\[ B(cn) = 2^m - 1 - \ell \quad n = 2^m + \ell \]
\[ C(cn) = \ell \]

\[ \ell \]

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Consider the special case:
\[ \alpha = 1, \beta = \gamma = 0 \]

\[ A(1) = 1 \]
\[ A(2n) = 2A(n) \quad \Rightarrow \quad A(2^m n) = 2^m \]
\[ A(2n+1) = 2A(n) \]

What values \((\alpha, \beta, \gamma)\), give us \(f(cn) = 1\)?

\[ f(1) = 1 = \alpha \quad (1, -1, -1) \]
\[ f(2n) = 2f(n) + \beta \]
\[ 1 = 2 \cdot 1 - 1 \quad \Rightarrow \quad \beta = -1 \]
\[ f(2n+1) = 2f(cn) + \gamma \quad \Rightarrow \quad \gamma = -1 \]

\[ A(n) - Bcn - Ccn = 1 \quad (1, 0, 1) \]

Try \(f(cn) = n\)

\[ f(1) = 1 = \alpha \]
\[ 2n = 2n + \beta \quad \Rightarrow \quad \beta = 0 \]
\[ 2n + 1 = 2n + \gamma \quad \Rightarrow \quad \gamma = 1 \]

\[ A(n) = 2^n \]

\[ A(n) - Bcn - Ccn = 1 \quad \Rightarrow \quad \ast \]

\[ A(cn) + C(n) = n \]
This is the “repertoire method” for solving recurrences.

1. Find settings for special parameters where we know the solution.

2. Combine the particular solutions to give the general solution.

Works often for linear recurrences.

**Shift Property:**

\[
J((b_m \ldots b_0)_2) = (b_{m-1} \ldots b_0 b_m)_2
\]

\[b_m = 1\]

Does this carry over to the generalized problem:

\[
f(i) = \alpha
\]

\[f(2n+j) = 2f(n) + \beta_j \quad j = 0, 1, n \geq 1\]

\[
f((b_m b_{m-1} \ldots b_1 b_0)_2)
\]

\[
= 2f((b_m b_{m-1} \ldots b_1)_2) + \beta_{b_0}
\]

\[
= 4f((b_m b_{m-1} \ldots b_2)_2) + 2\beta_{b_1} + \beta_{b_0}
\]

\[
f(i)
\]

\[
= 2^m \alpha + 2^{m-1} \beta_{b_{m-1}} + \ldots + 2\beta_{b_1} + \beta_{b_0}
\]
This works in general with $\beta_0 = \beta$, $\beta_1 = \gamma$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
<th>$(n)_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$2\alpha + \beta$</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>$2\alpha + \gamma$</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>$4\alpha + 2\beta + \beta$</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>$4\alpha + 2\beta + \gamma$</td>
<td>101</td>
</tr>
<tr>
<td>6</td>
<td>$4\alpha + 2\gamma + \beta$</td>
<td>110</td>
</tr>
<tr>
<td>7</td>
<td>$4\alpha + 2\gamma + \gamma$</td>
<td>111</td>
</tr>
</tbody>
</table>

$f(100) = f((1100100)_2)$

$\alpha = -1$

$\beta = -1$

$\gamma = +1$

One more generalization

$f(cj) = \alpha j$ $1 \leq j < d$

$f(dn + j) = (f(n)) + \beta j$ $0 \leq j < d$

$-21-$ $n \geq 1$
We start with number in radix \( d \) and end with them in radix \( c \).

\[
\phi \left( (b_m b_{m-1} \ldots b_1 b_0)_d \right) = (\alpha_{b_m} \beta_{b_{m-1}} \ldots \beta_{b_1} \beta_{b_0})_c
\]

\[
\begin{align*}
f(1) &= 34, & f(2) &= 5, & f(3n) &= 10fn + 76, \\
f(3n+1) &= 10fn + -2, & f(3n+2) &= 10fn + 8
\end{align*}
\]

\[
d = 3 \\
c = 10
\]

\[
f(19) = \phi \left( (201)_3 \right)
\]

\[
= (5 \ 76 \ -2)_{10}
\]

\[
= 500 + 760 - 2
\]

\[
= 1258
\]