Concrete Mathematics

$\underbrace{\text{Con}}_{\text{Continuous}}\underbrace{\text{crete}}_{\text{Discrete}}$

The mathematical basis for advanced Computer Science (minus the graph theory)

See syllabus

Recurrences

What should you already know?

Solving $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Reference: Cormen, Leiserson, Rivest
Introduction to Algorithms

Methods

a) Repeated substitution & inspection
b) Tree method (graphical)
c) Master Theorem

$$T(n) = a\, T\left(\frac{n}{b}\right) + f(n)$$

\# of subproblems of size $\frac{n}{b}$

recombination cost

Compare $n^{\log_b a}$ to $f(n)$

the bigger one wins, in case of a tie, multiply by $\lg n$

Master Theorem (Method)

Case 1: if $f(n) = \Theta(n^{\log_b a - c})$, $c > 0$

then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then

$$T(n) = \Theta(f(n) \lg n)$$
$$= \Theta(n^{\log_b a} \lg n)$$

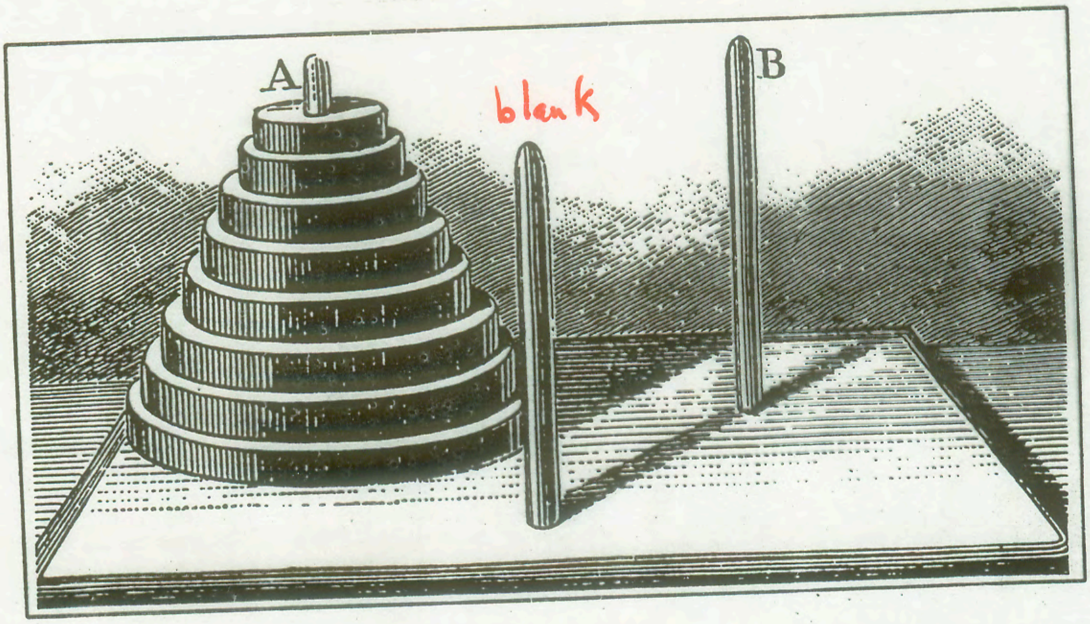
Case 3: if $f(n) = \Omega(n^{\log_b a + c})$, $c > 0$

and $a f(\frac{n}{b}) \leq c n$, $0 < c < 1$,

then $T(n) = \Theta(f(n))$

Some Problems

Tower of Hanoi



T_n = minimum number of moves required to move a tower of "n" disks

($T(n)$?)

By playing around

$T_1 = 1$ (duh!)

$T_2 = 3$ by inspection

With a tower of n, we must move n-1 from A to blank, biggest from A to B and then the n-1 from blank to B.

This means $T_n \le 2T_{n-1} + 1 \quad n > 0$

Can also argue

$$T_n \ge 2T_{n-1} + 1 \qquad n > 0$$

We must eventually move the largest disk, to do this the n-1 smaller must be on a single spindle, taking T_{n-1} in the best case. We then must move the disk to B and continue.

$$T_0 = 0$$

Note: Recurrences have two parts:

$T_0 = 0$ boundary condition

$T_n = 2T_{n-1} + 1 \quad n > 0$ (iteration)

Note: This does <u>not</u> fit into the master's method rubrik!

$$T(n) = 2T(n-1) + 1$$
$$= 2\,T\left(n \cdot \frac{n-1}{n}\right) + 1$$

a

b ?

no b must be constant

How do we solve this?

$T_0 = 0$

$T_1 = 1$

$T_2 = 2 \cdot 1 + 1 = 3$

$T_3 = 2 \cdot 3 + 1 = 7$

$T_4 = 2 \cdot 7 + 1 = 15$

$T_5 = 2 \cdot 15 + 1 = 31$

Computer Scientists should recognize the pattern:

$$T_n = 2^n - 1, \quad n \geq 0$$

This is a guess, must prove by induction.

$T_0 = 0 \qquad 0 = n_0$, the basis

Assume $T_{n-1} = 2^{n-1} - 1$

$$T_n = 2 \cdot T_{n-1} + 1 = 2 \cdot (2^{n-1} - 1) + 1$$
$$= 2^n - 2 + 1 = 2^n - 1 \qquad \square$$

Note : $U_0 = 1$

$U_n = 2\, U_{n-1}\,, \quad n > 0$

$$U_n = T_n + 1 \quad \Rightarrow \quad T_n = U_n - 1$$

$$T_0 + 1 = 1$$
$$T_n + 1 = 2\, T_{n-1} + 1 + 1$$
$$= 2\, T_{n-1} + 2 = 2\,(T_{n-1} + 1)$$

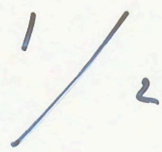
$$U_n = 2^n \quad \text{so} \quad T_n = 2^n - 1$$

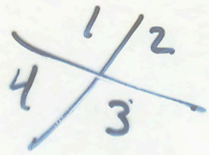
Second warmup problem
Lines in the plane

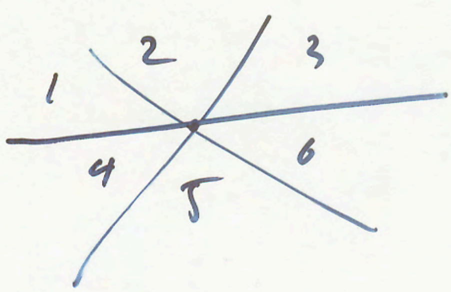
Problem posed by Steiner in 1826.

Compute L_n, the maximum number of regions the plane (pizza) can be cut by n straight lines

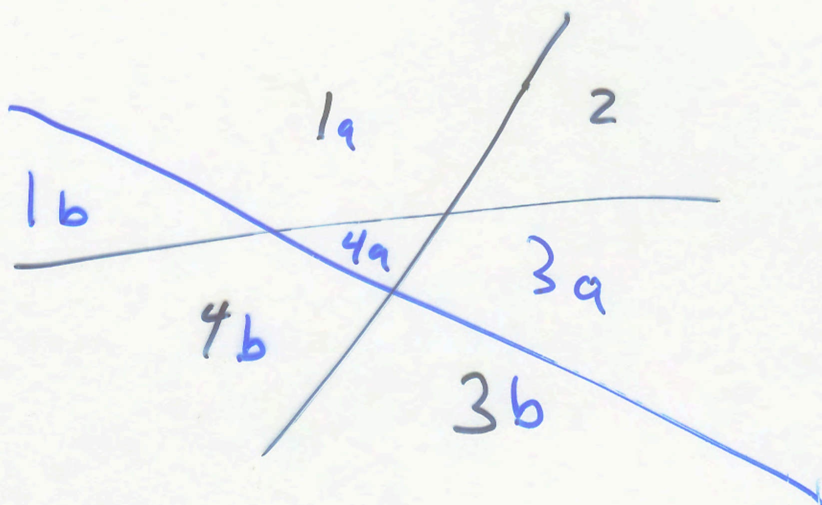
$L_0 = 1 \quad L_1 = 2 \quad L_2 = 4 \quad L_3 = ?$







can do better



$L_3 = 7$ note: above fails because the new line intersects the old lines at only a single point.

What if our construction always adds a line that intersects each of the other lines at least once. Doable? Yes by ensuring the new line is not parallel to the existing lines.

$$L_n = L_{n-1} + n \qquad n > 0$$

$$L_0 = 1 \qquad (B.C.)$$

$$L(n) = L\left(n \cdot \frac{n-1}{n}\right) + n$$

not constant

$L_0 = 1$

$L_1 = 1 + 1 = 2$

$L_2 = 2 + 2 = 4$

$L_3 = 4 + 3 = 7$ this is not so helpful

$$\begin{aligned} L_n &= L_{n-1} + n \\ &= L_{n-2} + n + (n-1) \\ &= L_{n-3} + n + (n-1) + (n-2) \\ &= L_0 + n + (n-1) + (n-2) + \dots + 1 \\ &= 1 + S_n \end{aligned}$$

$$S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$L_n = \frac{n(n+1)}{2} + 1$$

Proof: $L_0 = 1$ holds

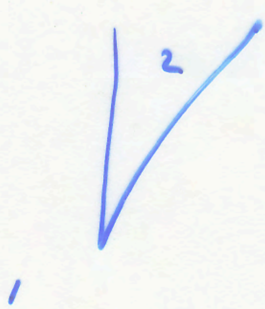
$$L_{n-1} = \frac{(n-1)(n-1+1)}{2} + 1 = \frac{n(n-1)}{2} + 1$$

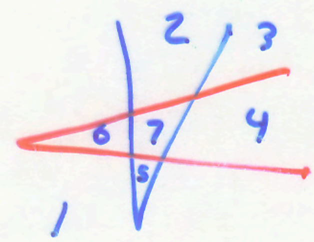
$$L_n = \frac{n(n-1)}{2} + 1 + n = \frac{n^2 - n}{2} + \frac{2n}{2} + 1$$

$$= \frac{n^2 + n}{2} + 1 = \frac{n(n+1)}{2} + 1$$

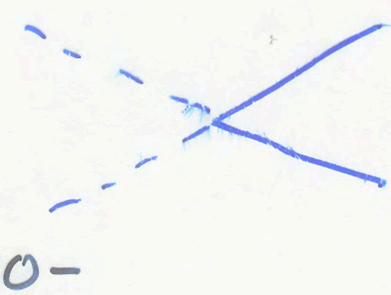
$\square$

Consider





$Z_1 = 2$ $Z_2 = 7$



This is like the previous problem but with two lines and merging.

$$Z_n = \underbrace{L_{2n}}_{\text{double lines}} - \underbrace{2n}_{\text{loss from merging}}$$

$$Z_n = \frac{2n(2n+1)}{2} + 1 - 2n$$

$$= 2n^2 + n + 1 - 2n$$

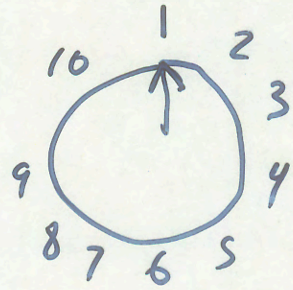
$$= 2n^2 - n + 1 \quad , n \geq 0$$

$$Z_n = \Theta(n^2)$$

$$L_n = \Theta(n^2)$$

not really surprising.

The Josephus Problem



eliminate in order 2, 4, 6, 8, 10, 3, 7, 1, 9 with 5 surviving.

$J(10) = 5$ is implied , $J(n) = \frac{n}{2}$?

n	1	2	3	4	5	6
$J(n)$	1	1	3	1	3	5

, <u>no</u>!

Since by convention we always start with 2, we always eliminate all the evens. Thus if we have $2n$ (an even number) of people

$$J(2n) = 2\,J(n) - 1$$

just solve on problem twice a small and re-number.

So $J(10) = 2J(5) - 1 = 2 \cdot 3 - 1 = 5$

checks out. Also

$$J(20) = 2J(10) - 1 = 2 \cdot 5 - 1 = 9$$
$$= 2(2J(5) - 1) - 1$$
$$= 4J(5) - 3 = 4 \cdot 3 - 3 = 9$$

Can prove (by induction):

$$J(5 \cdot 2^m) = 2^{m+1} + 1$$

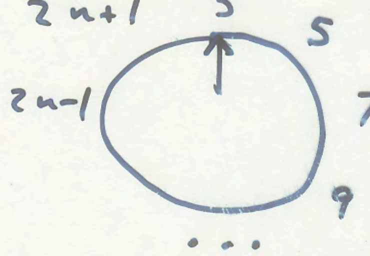
basis : $m = 0$ $\quad J(5) = 3 = 2^{0+1} + 1$ ✓

$$J(5 \cdot 2^m) = 2 \cdot J(5 \cdot 2^{m-1}) - 1$$
$$= 2 \cdot (2^m + 1) - 1$$
$$= 2^{m+1} + 2 - 1 = 2^{m+1} + 1$$

induction hypothesis

With $2n+1$ people (odd)

we wipe out first all the evens and then 1 :



Thus

$$J(2n+1) = 2J(n) + 1$$

(Check it yourself)

Combining we get the recurrence:

$$J(1) = 1 \qquad (B.C.)$$

$$\left.\begin{aligned} J(2n) &= 2J(n) - 1 \\ J(2n+1) &= 2J(n) + 1 \end{aligned}\right\} n \geq 1$$

Can use this as a "fast leap-ahead"

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Note the grouping by powers-of-two!

$$n = 2^m + l, \quad 0 \leq l < 2^m$$

↑ largest power-of-two in n

Seems $J(n) = 2l + 1$

Induction on m:

$m = 0 \qquad n = 2^0 + \ell$

$= 1 \qquad \ell = 0$

$$J(1) = 2 \cdot 0 + 1 = 1 \checkmark$$

Since the recurrence has two formulas, we must use both:

$$J(2^m + \ell) = 2\,J(2^{m-1} + \tfrac{\ell}{2}) - 1$$

$$= 2 \cdot \left(2 \tfrac{\ell}{2} + 1\right) - 1$$

even case

$$= 2\ell + 1$$

$$= 2\,J(2^{m-1} + \tfrac{\ell - 1}{2}) + 1$$

$$= 2 \cdot \left(2 \tfrac{\ell-1}{2} + 1\right) + 1$$

odd case

$$= 2 \cdot (\ell - 1) + 2 + 1$$

$$= 2\ell + 1$$

This is a closed-form solution.

Consider

$n = (b_m b_{m-1} \cdots b_1 b_0)_2$ in base 2

Note $b_m = 1$ (Why?)

$$\ell = (0\, b_{m-1}\, b_{m-2} \cdots b_1 b_0)_2$$
$$2\ell = (b_{m-1}\, b_{m-2} \cdots b_1 b_0\, 0)_2$$
$$2\ell+1 = (b_{m-1}\, b_{m-2} \cdots b_1\, b_0\, \underset{b_m}{1})_2$$

So $J((b_m \cdots b_0)_2) = (b_{m-1} \cdots b_0 b_m)_2$

is a one-bit left cyclic shift.

$n = 100 = (1100100)_2$

$J(n) = (1001001)_2 = 64 + 8 + 1 = 73$

$100 = 64 + \underset{\ell}{36}$ $\qquad 2\ell+1 = 72+1 = 73$

One oddity, $n = 13 = (1101)_2$

$J((1101)_2) = (1011)_2$; $J((1011)_2) = (0111)_2$

gets dropped

This is because $J(n) < n$.

If we iterate $J(\cdot)$ what happens?

"We squeeze the 0's out" and eventually get a fixed point $J(n) = n$ with $n = (1111\cdots1)_2 = 2^{r(n)} - 1$

$r(n) =$ pop-count(n) (# of 1's in binary n).

What about the original conjecture:

$$J(n) = \frac{n}{2} \quad \text{holds when?}$$

$$2\ell+1 = \frac{1}{2}(2^m+\ell) = 2^{m-1} + \ell/2$$

$$\frac{3}{2}\ell = 2^{m-1} - 1$$

$$\ell = \frac{1}{3}(2^m - 2)$$

If ℓ is an integer then $n = 2^m + \ell$ will solve this. $2^m - 2 \equiv 0 \pmod 3 \Rightarrow m$ odd

$110 = 6$ ✓
$1010 = 10$
$10010 = 18$ ✓
$100010 = 34$ ✓

m	ℓ	$n = 2^m + \ell$	$J(n) = \frac{n}{2}$	n (base 2)
1	0	2	1	10
3	2	10	5	1010
5	10	42	21	101010
7	42	170	85	1010101010

Generalize

$$f(1) = \alpha \qquad (1)$$
$$f(2n) = 2f(n) + \beta \qquad (-1)$$
$$f(2n+1) = 2f(n) + \gamma \qquad (1)$$

n	$f(n)$
1	α
2	$2\alpha + \beta$
3	$2\alpha + \quad + \gamma$
4	$4\alpha + 3\beta$
5	$4\alpha + 2\beta + \gamma$
6	$4\alpha + \beta + 2\gamma$
7	$4\alpha + \quad + 3\gamma$
8	$8\alpha + 7\beta$
9	$8\alpha + 6\beta + \gamma$

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

where here :

$$A(n) = 2^m$$

(*) $B(n) = 2^m - 1 - l$ $\qquad n = 2^m + l$

as before

$$C(n) = l$$

Consider the special case

$\alpha = 1 \ , \beta = \gamma = 0$

$\circledast$ $A(1) = 1$

$A(2n) = 2A(n) \implies A(2^m + l) = 2^m$

$A(2n+1) = 2A(n)$

what values, (α, β, γ), give us $f(n) = 1$?

$(1, -1, -1)$

$f(1) = 1 = \alpha$

$\left.\begin{array}{l} f(2n) = 2f(n) + \beta \\ 1 = 2 \cdot 1 - 1 \end{array}\right\} \beta = -1$ $\circledast$

$f(2n+1) = 2f(n) + \gamma \implies \gamma = -1$

$A(n) - B(n) - C(n) = 1$

$(1, 0, 1)$

try $f(n) = n$

$f(1) = 1 = \alpha$ $\circledast$

$2n = 2 \cdot n + \beta \implies \beta = 0$

$2n+1 = 2n + \gamma \implies \gamma = 1$

$\left.\begin{array}{ll} \circledast & A(n) = 2^m \\ \circledast & A(n) - B(n) - C(n) = 1 \\ \circledast & A(n) + C(n) = n \end{array}\right\} \implies \circledast$

This is the "repertoire method" for solving recurrences.

1. Find settings for special parameters where we know the solution.
2. Combine the particular solutions to give the general solution

Works often for linear recurrences.

Shift - Property:

$$J((b_m \cdots b_0)_2) = (b_{m-1} \cdots b_0 b_m)_2$$

$$b_m = 1$$

Does this carry over to the generalized problem:

$(\beta_0 = \beta,\ \beta_1 = \gamma)$

$$f(1) = \alpha$$
$$f(2n+j) = 2f(n) + \beta_j \qquad j = 0,1,\ n \geq 1$$

$$f((b_m b_{m-1} \cdots b_1 b_0)_2)$$
$$= 2f((b_m b_{m-1} \cdots b_1)_2) + \beta_{b_0}$$
$$= 4f((b_m b_{m-1} \cdots b_2)_2) + 2\beta_{b_1} + \beta_{b_0}$$
$$\cdots$$

f(1)

$$= 2^m \alpha + 2^{m-1}\beta_{b_{m-1}} + \cdots + 2\beta_{b_1} + \beta_{b_0}$$

This works in general with $\beta_0 = \beta$, $\beta_1 = \gamma$

n	$f(n)$	$(n)_2$
1	α	1
2	$2\alpha + \beta$	10
3	$2\alpha + \gamma$	11
4	$4\alpha + 2\beta + \beta$	100
5	$4\alpha \quad 2\beta + \gamma$	101
6	$4\alpha \quad 2\gamma + \beta$	110
7	$4\alpha + 2\gamma + \gamma$	111

$$f(100) = f((1100100)_2)$$

$$n = (1 \;\; 1 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0)_2 = 100$$

$$f(n) = +2^6 + 2^5 - 16 - 8 + 4 - 2 - 1 = 73$$

$\alpha \cdot 2^6$ $\quad \beta = -1$ $\quad \gamma = +1$

One more generalization

$$f(j) = \alpha_j \qquad 1 \le j < d$$

$$f(dn + j) = c\, f(n) + \beta_j \qquad 0 \le j < d$$

$n \ge 1$

We start with number in radix d and end with them in radix c.

$$f((b_m b_{m-1} \cdots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \cdots \beta_{b_1} \beta_{b_0})_c$$

$$f(1) = 34, \quad f(2) = 5, \quad \left.\begin{aligned} f(3n) &= 10f(n) + 76 \\ f(3n+1) &= 10f(n) + -2 \\ f(3n+2) &= 10f(n) + 8 \end{aligned}\right\} n \geq 1$$

$d = 3$

$c = 10$

$$\begin{aligned} f(19) &= f((201)_3) \\ &= (5\ 76\ {-2})_{10} \\ &= 500 + 760 - 2 \\ &= 1258 \end{aligned}$$